Self-fulfilling Fire Sales, Bank Runs and Contagion*

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Abstract

In a global-games framework, we endogenize bank asset fire sales by emphasizing asymmetric information: asset prices collapse because in banking crises, assets sold by illiquid banks can hardly be distinguished from those by insolvent banks. The lack of information makes runs and fire sales self-fulfilling and mutually reinforcing; it also generates financial contagion when banks have common risk exposures. The theoretical framework delivers several policy insights. (1) High capital holding can have unintended consequences on bank liquidity, because a run on a well-capitalized bank signals unusually high risk and exacerbates fire sales. (2) A regulator can improve financial stability by purchasing assets at a committed price. Such intervention resembles an asset purchase program and can break down the vicious cycle fueled by beliefs. Finally, (3) regulatory transparency involves a trade-off: while a favorable disclosure saves banks from illiquidity, acknowledging a crisis aggravates financial instability.

Keywords: Bank run, Global games, Asymmetric information, Capital, Regulatory transparency

JEL Classification: G01, G11, G21

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1 Introduction

The recent banking crisis highlights the risk of illiquidity. On the one hand, market liquidity evaporated and asset prices dropped sharply. On the other hand, as funding liquidity dried up, even well capitalized banks found it difficult to rollover their short-term debts and had to resort to central banks.

The two types of illiquidity closely link to each other. First, it has been well acknowledged that market illiquidity contributes to funding illiquidity. As market liquidity diminishes, potential fire-sale losses from early liquidation make creditors panic. Creditors can have coordination failures in rolling over their short-term debts and thus deprive a healthy financial institution of its funding. A bank can be solvent but illiquid: being able to repay in full its debts if no run happens, but being liquidated early if its creditors do not roll over their debts. The point has been emphasized by work like Morris and Shin (2000), Rochet and Vives (2004) and Goldstein and Pauzner (2005). However, the literature has ignored the feedback from bank runs to asset prices, treating separately two interconnected issues that in our opinion should be integrated.

Indeed, funding illiquidity also feeds market illiquidity. Bank runs can lead to fire-sales, depress asset prices, and in extreme cases, freeze up markets. As narrated by Acharya and Roubini (2009):

...the collapses on June 20, 2007, of two highly levered Bear Stearns-managed hedge funds that invested in subprime asset-backed securities (ABSs)...as the prices of collateralized debt obligations (CDOs) began to fall...lenders to the funds demanded more collateral...Merrill Lynch, seized $800 million of their assets and tried to auction them off. When only $100 million worth could be sold, the illiquid nature and declining value of the assets became quite evident.

The mutual reinforcement between market and funding illiquidity with the emergence of “liquidity spirals”, is first outlined by Brunnermeier and Pedersen (2009). Our model provides new insights into this issue by emphasizing coordination failures among creditors and asymmetric information in the secondary asset market.

We present a theoretical framework where asset fire-sales and bank runs/contagion happen in a self-fulfilling manner. When buyers of a bank’s assets are uninformed of a bank’s asset quality, observing a run will imply low asset values from buyers’ perspective. As the uninformed buyers cannot distinguish assets sold by solvent-but-illiquid banks from those by insolvent ones, such adverse selection will distort downwards their willingness to pay. As a
result, a solvent bank will not recoup a fair value for its assets on sale. The friction leads to a vicious circle. First, low asset prices fuel self-fulfilling bank runs: To avoid fire-sale losses caused by other creditors’ early withdrawals, a creditor has the incentive to withdraw funds from a solvent bank. Strategic complementarities can create successful runs and illiquid banks. Second, fire sales are self-fulfilling too. Out of the fear of low fire-sale prices, creditors run on a solvent bank and force early liquidation. Yet it is the run and liquidation, by pooling the solvent with the insolvent, that leads to the low fire-sale prices in the first place. In this sense, the creditors’ pessimistic expectation realizes itself. The self-fulfilling bank runs and fire sales intertwine and feed back into each other. Driven by the adverse selection, the whole banking crisis of “twin” illiquidity rises as a self-fulfilling prophecy. (See Figure 1 for an illustration.)

Figure 1: A banking crisis of “twin” illiquidity

For the financial system, contagion happens in a similar self-fulfilling manner except with one more ingredient—a common risk factor. As the uninformed buyers form rational expectations, they revise their expectation downwards of the common risk factor upon observing a bank run. The reduced expectation lowers their willingness to pay for other banks’ assets, which in turn precipitates runs in all other banks. It should be noted that such contagion (the run to other banks) again reflects the mutually reinforcing interaction between fire-sales and runs. Anticipating the declining asset prices due to buyers’ lower expectation of common risk factor, the creditors of other banks panic and run, and the run confirms the worsening expectation and leads to further distressed asset prices.

As a defining feature that distinguishes the current model from the literature, we have buyers’ beliefs, asset prices, bank runs, and contagion, all endogenous and jointly determined in
a rational expectation equilibrium. We prove that the equilibrium exists and is unique. These features of our model allow us to deliver several policy insights. In particular, we show that increasing bank capital and regulatory transparency can have unintended consequences, and refore challenge some conventional wisdom.

First, while our paper confirms that well capitalized banks have larger buffers against fire-sale losses, our analysis also reveals that once asset prices are endogenous, the situation is more complex and increasing capital also has unintended consequences on illiquidity and total credit risk. In particular, increasing bank capital can negatively affect asset prices via buyers’ beliefs. For an individual bank, buyers’ posterior beliefs on the bank’s asset value deteriorate when a run happens. And the deterioration is particularly strong when the bank maintains a high capital ratio. Because well capitalized banks are able to sustain large losses, if a run happens to such a bank, the bank’s losses must be unusually high. Therefore, given that a bank faces a run, buyers’ valuation of its assets decreases in its capital level. The low willingness to pay contributes to creditors’ coordination failure and makes the run more likely to happen in the first place. We show that in some extreme cases, increasing bank capital cannot reduce the risk of bank runs at all.

Second, our theoretical model confirms the effectiveness of asset purchase programs in promoting financial stability. In an asset purchase program where a regulator purchases bank assets at a committed price, the vicious cycle fueled by beliefs can be broken down. We argue that regulators hold more commitment power than other market participants, and the lack of commitment in ordinary asset buyers is at the very root of financial instability in this model. In particular, an ordinary asset buyer would behave according to her rational beliefs, and would avoid losses in every realized state. This can generate the vicious cycle discussed above because the buyer’s pessimistic belief can lead to negative market outcomes (e.g., more bank runs) which in turn justify itself. A regulator with commitment power, on the other hand, can resist such pessimistic belief updating. We show that even if the regulator has no better information than ordinary asset buyers, he can still break even and promote financial stability.

Finally, the information-based run and contagion links to the debate on regulatory transparency.¹ Our paper considers such transparency to be a double-edged sword. If the disclosed information reassures the asset buyers, illiquid banks will be saved. However, if the assistance program adds to pessimistic market inference, e.g., its size greater than expected, the assistance

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¹For instance, whether or not regulators should disclose information concerning their assistance programs.
program itself will be contagious. Once the severity of the problem is acknowledged, market participants further revise down the expected performance of all financial institutions’, leading to greater fire-sale losses and triggering illiquidity, even for healthy institutions. The fall of Bear Stearns was an interesting case in this aspect. As documented in Brunnermeier (2009):

> ...March 11, 2008, when the Federal Reserve announced its $200 billion Term Securities Lending Facility. ... However, some market participants might have (mistakenly) interpreted this move as a sign that the Fed knew that some investment bank might be in difficulty. Naturally, they pointed to the smallest, most leveraged investment bank with large mortgage exposure: Bear Stearns.

It was unclear whether Bear Sterns was truly insolvent or not. Yet because market participants believed the Fed was better informed and the action of Fed reflected that superior information, the attack began.

Our theoretical framework is related to the literature on bank runs and financial contagion. Since Diamond and Dybvig (1983) the literature is concerned with the financial fragility caused by runs. Following their seminal contribution there was a debate as to whether bank runs are due to pure panic or unfavorable information on banks’ fundamentals. The gap between the panic and fundamental view is bridged by the application of global games. Using the concept, papers such as Morris and Shin (2000), Rochet and Vives (2004) and Goldstein and Pauzner (2005) refine the multiple equilibria in Diamond and Dybvig (1983) and emphasize the role of early liquidation loss in causing bank runs: An extra buffer of cash flow is needed to reassure creditors and to prevent runs. Weak banks that fail to provide the extra buffer become “solvent but illiquid”. A limitation of the existing models is that they build on the simplifying assumption of exogenous fire-sale losses, so that the models ignore the reinforcing effects of runs on fire-sales. In contrast, the current paper explores the relationship: As it is difficult to distinguish the illiquid banks from those insolvent ones, the adverse selection causes the low asset prices and fire-sale losses. 

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2It should be mentioned that some papers also consider the positive role of bank run as disciplinary device: Calomiris and Kahn (1991) and Diamond and Rajan (2001).


5While the current paper justifies the low asset price by informational frictions, low asset prices can also be explained by fixed short-term cash supply—the cash-in-the-market argument pioneered by Shleifer and Vishny (1992) and Allen and Gale (1994).
A natural corollary of assuming an exogenous fire-sale price is that funding liquidity risk will be always reduced by higher capital, because the returns generated on capital add to the buffer against fire-sale losses. With endogenous fire-sale prices the current paper takes a broader view: while acknowledging the buffer effect of capital, we point out that greater capital can also contribute to illiquidity via buyers’ pessimistic inference.

Predicting an interaction between market liquidity and funding liquidity, our model is most closely related to Brunnermeier and Pedersen (2009), who emphasize a haircut constraint on a speculator that supplies liquidity to a financial market with limited participation. In their model, asset prices are volatile because there is an asynchronization between selling and buying. This paper differs from theirs in two aspects. First, the funding liquidity risk rises as a result of equilibrium bank runs caused by the wholesale creditors’ coordination failures. Second, this paper emphasizes the asymmetric information on asset qualities, and how such adverse selection causes asset illiquidity.

In our paper, contagion is generated not only by the actual realization of common risk factor but also by its perception: A bank failure casts shadow on the perceived common risk factor; and the created negative informational externalities affect all the other banks. This observation is mostly related to the literature of information contagion, as exemplified by Acharya and Thakor (2011) and Oh (2012). Compared to the existing work, the current paper emphasizes the self-fulfilling nature of such contagion and the two-way feedback between runs and fire-sales.

On the application to capital requirements, the paper relates to a few papers that show increased capital requirements can increase bank risk. Martinez-Miera (2009) argues that equity increases banks’ cost of funding, which leads to higher loan rates and spurs risk-taking by borrowers. As a result, banks’ portfolio risk rises passively. Hakenes and Schnabel (2007) argue that a higher capital requirement erodes charter value and induces banks’ active risk taking; when the higher capital requirement decreases credit supply, it also leads to borrower risk-taking via a hike in loan rate. What all these papers have in common is that they all focus on solvency risk. To the best of our knowledge, the current study is the first to show capital can contribute to illiquidity, contagion and systemic risk.

The discussion on disclosure policy is most related to several recent papers on the instability consequences of public signals: Morrison and White (2010) is concerned that a public bailout can reveal regulatory deficiency and make market participants lose their confidence in all other banks under the same regulation. Dang, Gorton, and Holmström (2010) shows that a public signal makes debt-like securities information sensitive, could otherwise increase adverse selection. Wang (2013) empirically documents that after individual banks were identified in Trouble Asset Relief Program (TARP), bank run probabilities, as reflected in CDS spread and stock market abnormal returns, rose dramatically, an outcome the author attributes to the bad news nature of public bailout. Our paper abstracts from specific policy announcements and shows that as long as market participants believe the regulator is better informed, any regulatory action and announcement concerning banks’ common risk exposure may generate financial contagion.

The paper proceeds as follows. Section 2 lays out the model. Section 3 presents the baseline bank-run model under asset market adverse selection and endogenous fire-sale price. With only one bank and one state, the baseline model allows us to discuss the first policy issue that whether higher capital can lead to greater illiquidity risk and total credit risk. In Section 4, we analyze contagion in the full fledged model with two banks and two states. We are able to address the second policy issue that whether regulators should disclose information on aggregate states. In Section 5, we discuss briefly the implications for other related policy issues such as liquidity requirements and lender of last resort policies. Section 6 concludes.

2 Model setup

We consider a three-date \((t = 0, 1, 2)\) economy with two banks. At \(t = 0\), banks are identical. Each of them holds a unit portfolio of long-term assets, and finances them with equity \(E\), retail deposits \(F\), and short-term wholesale debts \(1 – E – F\). There are two groups of active players: banks’ wholesale creditors and uninformed buyers of banks’ assets. Both groups of players are risk neutral. We assume that retail deposits are fully insured so that depositors act only passively. Since their claims are risk free, the depositors will always hold their claims to maturity, and demand only a gross risk-free rate which we normalize to 1. We also assume that the financial safety net is provided to banks free of charge. We consider banks as contractual arrangements among claim holders, designed to fulfil the function of liquidity and maturity

\[\text{It should be emphasized that all results of the current paper can be generalized to a N-bank case.}\]
Therefore, banks in our model are passive, with given loan portfolios and liability structures.

Banks’ wholesale debts are risky, demandable, and raised from a continuum of creditors. Provided that a bank does not fail, a wholesale debt contract promises a gross interest rate \( r_D > 1 \) at \( t = 2 \), and \( qr_D \) if a wholesale creditor withdraws early at \( t = 1 \). Here \( q < 1 \) reflects the penalty for the early withdrawal. A bank run occurs if a positive mass of wholesale creditors withdraw funds from their bank at \( t = 1 \). For the ease of future exposition, we denote by \( D_1 \) the total amount of debts a bank needs to repay at \( t = 1 \) if all wholesale creditors withdraw early, and by \( D_2 \) the total amount of debts a bank needs to repay at \( t = 2 \) if no wholesale creditor withdraws early.

\[
D_1 \equiv (1 - E - F)qr_D \\
D_2 \equiv (1 - E - F)r_D + F
\]

A bank’s portfolio generates a random cash flow \( \tilde{\theta} \) at \( t = 2 \). For simplicity, we assume that \( \tilde{\theta} \) follows a uniform distribution on \([\underline{\theta}, \bar{\theta}]\), and the random cash flows of the two banks are independent and identically distributed. Subscript \( s \) denotes the realization of an aggregate state that affects both banks. There are two possible states, \( G \) and \( B \) (e.g., housing market boom or bust), and the two states occur with an equal probability. With \( \underline{\theta}_G > \underline{\theta}_B \), State \( G \) is more favorable than State \( B \). Therefore, the value of a bank’s assets is not only affected by its idiosyncratic risk (the realizations of \( \tilde{\theta} \)) but also by the aggregate risk \( s \). On the other hand, \( \bar{\theta} \) is assumed to be the same across states. This reflects the fact that banks hold mostly debt claims whose highest payoffs are capped by their face values. We further make the following three assumptions on parameters.

\[
D_2 > \theta_s \quad (1) \\
(\bar{\theta}_B + \bar{\theta})/2 > D_2 \quad (2) \\
F > D_1 \quad (3)
\]

As \( D_2 \) denotes a bank’s total debt obligation at \( t = 2 \), inequality (1) states that there is a positive probability of bankruptcy in both states. Inequality (2) states that, in the absence of bankruptcy

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8This view can be traced back to Diamond and Dybvig (1983).
cost, even if the realization of the state is unfavorable, the expected cash flow of a bank’s asset is still greater than its debt obligations, so that bank lending is viable. Finally, inequality (3) states that a bank’s retail debts exceed its wholesale debts, which is a realistic scenario and helps to simplify the analysis of bank run games.\(^9\) We also assume that bankruptcy costs are sufficiently high such that once a bank fails, its residual value drops to zero.

Banks’ assets are long-term, taking two periods to mature. In particular, we assume that at \(t = 1\) the assets cannot be physically liquidated. Therefore, if a wholesale run happens, to meet the liquidity demand, a bank has to financially liquidate its assets in a secondary asset market, and sell them to outside asset buyers. As early liquidation is costly in this model, a bank will sell its assets if and only if it faces a bank run.

### 2.1 Secondary asset market

Potential buyers in the secondary asset market are uninformed: they are unable to observe either the aggregate state \(s\) or any bank’s cash flow \(\theta\). Yet, they can observe the number of bank runs, and based on the observable outcome, form rational expectations about the quality of assets on sale. In this two-bank setup, there are three distinctive outcomes from the buyers’ perspective, i.e., the number of bank runs \(N = 0, 1,\) or \(2\).

We assume the following sequential moves between asset buyers and wholesale creditors. Asset buyers first post a price scheme \(P = (P_1, P_2)\), and offer to purchase bank assets on sale at price \(P_1\) when the number of bank runs \(N = 1\), and \(P_2\) when \(N = 2\). Having observed the price scheme, wholesale creditors play a bank run game, making their individual decisions simultaneously on whether to withdraw their funds early. In case that any bank run happens, transactions take place at the offered price, and assets are transferred to buyers.

The price scheme \(P\) is complete in the sense that an asset price is specified for each distinctive outcome of bank run games where bank assets are on sale. Depending on the number of runs observed, the prices that buyers offer can differ. In fact, in the absence of commitment power, the asset buyers’ decisions need to be time consistent so that they will not revoke their posted price after the outcomes of bank run games are revealed. As a result, the price \(P_1\) and \(P_2\) will have to reflect buyers’ posterior beliefs on asset qualities. As buyers form different

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\(^9\)It should be emphasized that the condition is more than a technical assumption. It is realistic in the sense that despite of the rapid growth of wholesale funding, most of commercial banks and bank holding companies are still financed more by retail deposits than wholesale debts.
posterior beliefs when observing different numbers of bank runs, their offered prices will vary with the number of bank runs.

The asset market is assumed to be perfectly competitive, and the buyers compete in the price schemes that they offer. In equilibrium, based on their posterior beliefs, the asset buyers should perceive themselves breaking even in expectation when purchasing bank assets at their posted prices. As the buyers make time-consistent decisions and do not revoke their offers, they must make no loss for any realized number of bank runs.

2.2 Bank run game

The demandable nature of wholesale debts allows creditors to withdraw their funds before a bank’s assets mature, which will force the bank to liquidate its assets prematurely. When assets are sold for less than their fundamental values, there will be an early liquidation loss, or an asset fire sale. While the creditors who withdraw early can avoid suffering from the fire sale, those who do not withdraw will receive zero payoffs if the bank fails. As a result, creditors’ actions to withdraw display strategic complementarities, and it can be in the interest of all creditors to run on a bank that is otherwise solvent.

A bank run game of complete information can have two strict equilibria that all creditors withdraw from the bank, and that nobody withdraws. To refine the equilibria, we take the global-games approach pioneered by Carlsson and Van Damme (1993) and study games with incomplete information, where common knowledge on $\theta$ does not exist among creditors. We assume that at the beginning of $t = 1$, both aggregate risk (State $s$) and idiosyncratic risk (cash flow $\theta$) have been realized, but the information is not fully revealed to players. For a given bank, each individual creditor only privately observes a noisy signal $x_i = \theta + \epsilon_i$. The noise $\epsilon_i$ is drawn from a uniform distribution with a support $[-\epsilon, \epsilon]$, where $\epsilon$ can be arbitrarily small. Based on their private signals, the creditors play a bank-run game with each other. Each of the creditors has two possible actions: to wait until maturity or to withdraw early, and follows a threshold strategy: to withdraw early if and only if their individual private signal is lower than a critical level $\hat{x}$. In this two-bank setup, we also assume that each creditor holds claims in both banks, and observes independent noisy signals for both banks’ cash flows.

The maturity mismatch between banks’ liabilities and assets, together with potential asset fire sales, exposes banks to the risk of runs. In particular, a run and premature liquidation at $t = 1$ can cause failure to a bank that is otherwise solvent at $t = 2$. In order to reassure its
creditors not to withdraw early, a bank has to be more than merely solvent, and should be able to absorb potential fire-sale losses. This implies a critical cash flow $\hat{\theta} > D_2$ for a bank to survive a run. The distance between $\hat{\theta}$ and $D_2$ provides a measure of financial instability. Moreover, a lower asset price implies greater fire-sale losses, and a higher critical cash flow $\hat{\theta}$ for a bank to survive a run.

Given our assumption that bankruptcy costs result in zero residual value, if a bank is to fail at $t = 1$, a wholesale creditor will receive zero payoff whether he withdraws early or not. In this case of indifference, we assume that the creditor will always withdraw. One justification can be that wholesale creditors receive arbitrarily small reputational benefits by running on a bank that is doomed to failed.\textsuperscript{10}

2.3 Asymmetric information on cash flow $\theta$

As asset buyers are intelligent, they can solve creditors’ bank run game and form rational beliefs on the qualities of assets on sale. In particular, they know that a bank will be forced into an asset sale if and only if its cash flow is below $\hat{\theta}$. However, the lack of more detailed information makes solvent banks (those with $D_2 \leq \theta < \hat{\theta}$) indistinguishable from the insolvent ones (those with $\theta < D_2$). As an equilibrium asset price reflects only the average quality of assets on sale, a bank with cash flow $\theta$ greater than the price but less than $\hat{\theta}$ will face an asset fire sale.

As a lower asset price pushes $\hat{\theta}$ upwards, there will be two-way feedback between asset fire sales and bank runs. When asset buyers offer a low price for a bank’s assets, a run is triggered, which generates the pooling of assets, and thus fully justifies the low asset price offered in the first place. As a result, both fire sales and bank runs occur in a self-fulfilling manner.

2.4 Belief updating on State $s$

While asset buyers hold a prior belief that State $B$ and $G$ occur with an equal probability, after observing any bank runs, they update their beliefs according to Bayes’ rule and consider State $B$ to be more likely. The pessimistic belief updating can lead to financial contagion. In particular, a bank may face no runs if the other bank does not face a run, but will if the other one does. This defines financial contagion in our model.

\textsuperscript{10}For more detailed discussion on this assumption, please see Rochet and Vives (2004).
In the current model, financial contagion is self-fulfilling too. When observing more bank runs, asset buyers infer State $B$ to be increasingly likely and reduce their offered asset prices accordingly. The fear of increased liquidation losses makes wholesale creditors panic even more, and leads to simultaneous bank runs in the first place.

2.5 Timing

The timing of the model is summarized in Figure 2. Events at $t = 1$ take place sequentially.

![Figure 2: Timing of the game](image)

<table>
<thead>
<tr>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Banks are established, with their portfolios and liability structures as given.</td>
<td>1. $s$ and $\theta$ are realized. 2. Asset buyers post a price scheme.</td>
<td>1. Returns become public. 2. Remaining obligations are settled.</td>
</tr>
<tr>
<td>3. For each bank that they lend to, creditors receive private noisy signals about the bank’s cash flow $\theta$, and decide to run or not. 4. After observing the number of bank runs, buyers purchase assets at the quoted price.</td>
<td></td>
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3 Self-fulfilling bank runs and fire sales

Depending on the realization of $\hat{\theta}$, the model can have two types of equilibria: one type with bank runs, and the other without. The market equilibrium with bank runs consists of two parts. First, the bank run games feature threshold equilibria. That is, when $N$ runs happen and bank assets are sold for an equilibrium price $P^*_N$, a bank will experience a run if and only if the bank’s cash flow is lower than a unique threshold $\theta^*_N \equiv \hat{\theta}(P^*_N)$, $N \in \{1, 2\}$. Second, the competitive asset market is in a rational expectations equilibrium. That is, asset buyers form a rational belief about the quality of assets on sale based on the number of bank runs $N$. In particular, they anticipate $\theta < \hat{\theta}(P^*_N)$, and Bayesian update their beliefs on State $s$. According to such posterior beliefs, asset buyers who purchase bank assets at an equilibrium price $P^*_N$ should perceive themselves breaking even in expectation. Moreover, the buyers should find themselves unable to profitably deviate from bidding $P^*_N$. 

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**Definition.** Denote $\hat{\theta}(P_N)$ the threshold equilibrium of the bank run game for a given asset price $P_N$; and $P_N(\hat{\theta})$ the price scheme by which asset buyers break even in expectation for a given $\hat{\theta}$ and their rational beliefs about $\theta$ and $s$. The equilibrium of the model is defined by equilibrium critical cash flows $\theta_N^* = \hat{\theta}(P_N^*), N \in \{1, 2\}$, and an equilibrium asset price scheme $P^* = (P_1^*, P_2^*)$ with $P_N^* = P_N(\theta_N^*)$. The combination of $\theta_N^*$ and $P_N^*$ is such that: when there are $N$ bank runs in the economy, (1) a successful bank run happens if and only if the bank’s cash flow is lower than $\theta_N^*$; (2) the competitive asset market is in a rational expectations equilibrium, where asset buyers form rational beliefs about State $s$ and the quality of assets on sale. Based on their posterior beliefs, the buyers perceive themselves making zero profit in expectation by purchasing bank assets at $P_N^*$ and cannot make profitable deviation.

It takes four steps to obtain the equilibrium.

- First, we show that equilibrium asset prices $P_N^*$ cannot be lower than $D_1$ or higher than $D_2$ (subsection 3.1). This restricts the set of candidate equilibria and will facilitate the solution of bank run games.

- Second, solving the model using backward induction, we start with creditors who move last and solve the bank run game using the concept of global games. For a given asset price $P_N \in (D_1, D_2)$, we derive a unique critical cash flow $\hat{\theta}(P_N)$, so that a bank run will happen if and only if the bank’s cash flow $\theta < \hat{\theta}(P_N)$ (subsection 3.2).

- Third, we characterize asset buyers’ posterior beliefs on asset qualities when $N$ bank runs occur. In particular, they expect only those assets with quality $\theta < \hat{\theta}(P_N)$ to be on sale, and update their beliefs about State $s$ using Bayes’ rule. It should be emphasised that the buyers’ rational beliefs are functions of asset prices that they offer (subsection 3.3).

- Finally, we solve for the equilibrium of the model by examining equilibrium asset price schemes. As asset buyers offer different prices given different numbers of bank runs, we solve for equilibrium prices $P_N^*$ for each $N \in \{1, 2\}$. For $N$ observed bank runs, in a competitive equilibrium, $P_N^*$ should be equal to the expected asset quality based buyers’ posterior beliefs (subsection 3.3).

To illustrate the main intuition behind the feedback between bank runs and fire sales, we present in subsection 3.4 a simplified version of the model where there is only one state so that asset buyers cannot update their beliefs on State $s$. This simplification allows us to derive a closed-form solution to our model, and is sufficient to generate some interesting result such as
unintended liquidity consequences of bank capital. The full-fledged model with different states and asset buyers’ belief updating on $s$ is analyzed in section 4.

3.1 Restricting the set of candidate equilibria

For an equilibrium price cannot be negative, a candidate equilibrium price $P_N^*$ can only fall into one of three regions, $0 \leq P_N^* \leq D_1$, $D_1 < P_N^* < D_2$, and $P_N^* \geq D_2$. We discuss the existence and uniqueness of equilibrium for each of the three regions, and show that any equilibrium price $P_N^*$ cannot be lower than $D_1$, nor greater than $D_2$ provided that $F > D_1$.

Suppose $P_N^* \geq D_2$. Then, for any bank with $\theta \in [D_2, \bar{\theta}]$, it is suboptimal for its wholesale creditors to withdraw early. This is because with $P_N^* \geq D_2$, an asset sale at $t = 1$ will not hurt the bank’s capability to repay its liabilities at either $t = 1$ or $t = 2$. As a result, by running on the bank, a creditor will only incur the penalty for early withdrawal. This implies that whenever a run happens, it must be the case that the bank is fundamentally insolvent with $\theta < D_2$. Therefore, the highest asset quality that buyers can expect is $D_2$, with the expected quality strictly lower than that. As asset buyers break even and pay a price equal to the expected quality, the price that the buyers are willing to pay must be strictly smaller $D_2$. This contradicts the presumption $P_N^* \geq D_2$.

Now, suppose $P_N^* \leq D_1$. Then, a bank with $\theta \in [\underline{\theta}_s, D_2]$ will for sure fail, either because sufficiently many creditors run at $t = 1$, or because of fundamental insololvency at $t = 2$. Under the assumption that wholesale creditors run on banks that are doomed to fail, we know that successful runs must happen to those banks with $\theta \in [\underline{\theta}_s, D_2]$. This implies that the expected quality of assets on sale is at least $(\underline{\theta}_b + D_2)/2$. As asset buyers only break even in equilibrium, the price they offer must be greater than that. Therefore, we have $P_N^* > (\underline{\theta}_b + D_2)/2 > D_2/2$. By the definitions of $D_1$ and $D_2$, we further have $D_2/2 = [(1 - E - F)\gamma D + F]/2 > [(1 - E - F)\gamma r_D + F]/2 = (D_1 + F)/2$, which is in turn greater than $D_1$, provided $F > D_1$. Again, this contradicts the presumption $P_N^* \leq D_1$. We summarize these results in Lemma 1.

**Lemma 1.** An equilibrium asset price cannot less than or equal to $D_2$. And an equilibrium asset price cannot be greater than or equal to $D_1$ either, provided $F > D_1$. 

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3.2 Threshold equilibrium for bank run games

We solve the model by backward induction, and start with the subgame of bank runs. We show that for a given price $P_N \in (D_1, D_2)$ the bank run game has a unique threshold equilibrium characterized by a critical cash flow $\hat{\theta}(P_N)$. A successful bank run happens if and only if the bank’s cash flow is lower than $\hat{\theta}(P_N)$.

To solve for the optimal strategy of creditors, we first derive their payoffs for action “wait” and “withdraw” as functions of the number of other creditors who withdraw from the bank. Denote by $L \in [0, 1]$ the fraction of creditors who withdraw from the bank at $t = 1$. A bank that faces a total withdrawal of $LD_1$ can meet the demand for liquidity with a partial liquidation by selling a $f$ fraction of its assets.\(^\text{11}\)

\[
f = \frac{LD_1}{P_N} < 1
\]  

After liquidating $f$ fraction of its assets, the bank will fail at $t = 2$ if and only if the value of its remaining assets $(1 - f)\theta$ is lower than its remaining liabilities $F + (1 - L)(1 - E - F)r_D$. That is,

\[
(1 - f)\theta \leq F + (1 - L)(1 - E - F)r_D.
\]  

Thus, a bank will fail at $t = 2$ if and only if the fraction of creditors’ withdrawal exceeds a threshold $L^c$.

\[
L \geq \frac{P_N(\theta - F - (1 - E - F)r_D)}{(q\theta - P_N)(1 - E - F)r_D} = \frac{P_N(\theta - D_2)}{[\theta - P_N/q]D_1} \equiv L^c.
\]  

Such a $t = 2$ failure happens because the partial early liquidation incurs a cost of fire sale. When a sufficiently large number of creditors withdraw and the bank is forced to liquidate a significant share of assets prematurely, the remaining assets will not generate sufficient cash flows to meet the remaining liabilities. The creditors who withdraw early at $t = 1$ therefore can impose negative externalities on creditors who choose to wait.

Depending on the amount of early withdrawals $L$, a creditor’s payoffs of playing withdraw or stay are tabulated as follows.

<table>
<thead>
<tr>
<th>$L \in [0, L^c)$</th>
<th>$L \in [L^c, 1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>withdraw</td>
<td>$qr_D$</td>
</tr>
<tr>
<td>stay</td>
<td>$r_D$</td>
</tr>
</tbody>
</table>

\(^{11}\)Here $f < 1$ is guaranteed by $P_N > D_1$ and $L \leq 1$. Note that three factors contribute to a high fraction of asset liquidation: (i) a large number of early withdrawals, (ii) low market price $P$ for assets on sale, and (iii) a high level of wholesale debts.
Note that if a creditor withdraws, his payoff will always be $W_{\text{run}}(L) = q r_D$. Instead, if he waits, his payoff depends on the action of other creditors.

$$ W_{\text{wait}}(L) = \begin{cases} r_D & L \in [0, L^c] \\ 0 & L \in [L^c, 1] \end{cases} $$

Defining the difference between the creditor’s payoffs of withdraw and stay as $D W(L) \equiv W_{\text{run}}(L) - W_{\text{wait}}(L)$, one has

$$ DW(L) = \begin{cases} -(1 - q) r_D & L \in [0, L^c] \\ q r_D & L \in [L^c, 1] \end{cases} $$

The strategic complementarity is clear: when a sufficient large number of other creditors choose to withdraw ($L > L^c$), a wholesale creditor receives better payoff is better by withdrawal than to wait. In fact, when there is complete information on $\theta$, the bank run game has two equilibria in which either all creditors withdraw or all creditors wait. We refine the multiple equilibria using the technique of global games.

The analysis follows a standard global games approach. We give here the outline of the proof, and interested readers can refer to Appendix A for full details. First, we establish the existence of a lower dominance region $[\theta_L, \theta^l]$, where and it is a dominant strategy for all wholesale creditors to withdraw early, independent of the private signal that they receive. Similarly, we show there exists an upper dominance region $[\theta^u(P_N), \bar{\theta}]$, where it is a dominant strategy for all creditors to wait. For the intermediate range $\theta_L < \theta < \theta^u(P_N)$, a creditor’s payoff depends on the actions of other creditors. So, as a second step, we characterize a creditor $i$’s ex-post belief about the other creditors’ actions, conditional on his private signal $x_i = \theta + \epsilon_i$. The belief is a conditional distribution of $L$. The creditor then choose his optimal action based on the ex-post belief and payoff function $D W(L)$. Finally, for the limiting case where the noise of the signal approaches zero, we obtain a unique threshold

$$ \hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - q D_1 / P_N} \quad (7) $$

such that a successful bank run will happen if and only if the bank’s cash flow $\theta < \hat{\theta}(P_N)$. The results are summarized in Proposition 1.

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12 One can derive the upper and lower bounds explicitly and show that $\theta_L = D_2$ and $\theta^u(P_N) = \frac{F}{1 - D_2 / P_N}$. 

**Proposition 1.** For a secondary market asset price \( P_N \in (D_1, D_2) \), the bank run game has a unique threshold equilibrium: a successful run occurs to a bank if the bank’s cash flow fall below a critical level \( \hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1/P_N} \).

Proof. See Appendix A. \( \square \)

Expression (7) establishes a one-to-one correspondence between the asset price \( P_N \) and the critical cash flow \( \hat{\theta}(P_N) \). Note that the critical cash flow \( \hat{\theta}(P_N) \) is decreasing in \( P_N \). A lower asset price makes successful bank runs more likely.

### 3.3 Asset market equilibrium

The uninformed asset buyers observe neither \( \theta \) nor State \( s \), but they can form rational beliefs about the quality of asset on sale. First of all, they anticipate the threshold equilibrium for the bank run game to be characterized by \( \hat{\theta}(P_N) \). Therefore, when \( N \) bank runs happen, the asset buyers form a rational belief that only those assets of quality \( \theta < \hat{\theta}(P_N) \) will be on sale. Second, the asset buyers also update their beliefs about State \( s \) using Bayes’ rule. We denote \( \omega^G_N(\hat{\theta}(P_N)) \) the buyers’ posterior belief that \( s = G \) when the observed number of bank runs equals \( N \), and \( \omega^B_N(\hat{\theta}(P_N)) \) the posterior belief for \( s = B \). It should be emphasized that the posterior beliefs depend on buyers’ offered price \( P_N \).

Note that two factors can contribute to asset fire sales. First, conditional on a bank run has happened, the cash flow of the bank must be lower than \( \hat{\theta}(P_N) \). The buyers face an adversely selected asset pool in the sense that only those banks with low cash flow will be forced into asset sales. Second, any observed bank runs also indicate that \( s = B \) is more likely. This further reduces the expected quality of assets on sale, which in turn reduces buyers’ willingness to pay.

When the asset market is perfectly competitive, an equilibrium asset price must satisfy two conditions. First, based on their rational expectations about \( \theta \) and \( s \), the buyers should make zero expected profit by purchasing bank assets at the posted price. In other words, when there are \( N \) bank runs, an equilibrium asset price \( P_N^* \) equals the expected asset quality.

\[
P_N^* = \mathbb{E}[\theta|\theta < \hat{\theta}(P_N)] = \omega^G_N(\hat{\theta}(P_N^*)) \frac{\theta_G + \hat{\theta}(P_N^*)}{2} + \omega^B_N(\hat{\theta}(P_N^*)) \frac{\theta_B + \hat{\theta}(P_N^*)}{2} \quad (8)
\]

Second, a buyer should not be able to make profitable deviation by unilaterally bidding a higher price. Therefore, their expected net payoff, \( \mathbb{E}[\theta|\theta < \hat{\theta}(P_N), N] - P_N \), should not increase in \( P_N \).
The equilibrium has a fixed-point representation: \( P_N^* \) should be a fixed point for function \( E[\theta|\theta < \hat{\theta}(P_N), N] \). We show that for each \( N \in \{1, 2\} \), the fixed-point equilibrium exists and is unique. We also verify that the equilibrium is stable in the sense that a buyer cannot profitably deviate by unilaterally bidding a higher price.

### 3.4 A baseline model

The feedback between a bank run and an asset fire sale can be examined without different aggregate states. Therefore, to illustrate the main intuition, we analyze a baseline case of our model with \( \theta_B = \theta_G = \theta \). As buyers do not update their beliefs about State \( s \), their posted price scheme will consist of only one unified price \( P \). For this baseline model, we denote market equilibrium by \( \{\theta_e, P_e\} \), and obtain closed-form solutions.

As discussed, intelligent asset buyers can solve the subgame of bank runs and anticipate only those assets of quality \( \theta < \hat{\theta}(P) \) to be on sale. On the other hand, when the asset market is in a competitive equilibrium, asset buyers who purchase banks’ asset at the posted price should break even in expectation. Given their belief \( \theta \sim U(\underline{\theta}, \hat{\theta}(P)) \), a candidate equilibrium price \( P_e \) must satisfy the following zero-profit condition.

\[
P_e = \frac{\hat{\theta}(P_e) + \theta}{2}
\]  

With \( \hat{\theta}(P) \) derived in equation (7), we can write the condition explicitly.

\[
P_e = \frac{1}{2} \left( \frac{D_2 - D_1}{1 - qD_1/P_e} + \theta \right)
\]  

Equation (10) has one and only one root in interval \((D_1, D_2)\). We obtain the following closed-form solution of equilibrium asset price \( P_e \).

\[
P_e = \frac{(D_2 - D_1) + 2qD_1 + \theta + \sqrt{[(D_2 - D_1) + 2qD_1 + \theta]^2 - 8qD_1\theta}}{4}
\]  

For \( P_e \) to be an equilibrium, asset buyers should not have profitable deviation by unilaterally bidding a higher price than \( P_e \). That is, a buyer’s expected payoff, \( E[\theta|\theta < \hat{\theta}] - P \), should not

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\(^{13}\)Details can be found in Appendix B.1.
increase in $P$. In the baseline model, the asset buyers’ expected payoff takes the form

$$\pi(P) = \frac{1}{2} \left( \frac{D_2 - D_1}{1 - qD_1/P} + \theta \right) - P.$$ 

For $P > D_1$, the expected payoff monotonically decreases in $P$.

$$\frac{d\pi(P)}{dP} = -\frac{qD_1(D_2 - D_1)}{2(1 - \frac{q}{4}D_1)^2P^2} - 1 < 0$$

From (10), the equilibrium asset price is such that $\pi(P_e) = 0$, an asset buyer will earn negative profit if unilaterally bidding a higher price $P > P_e$. Intuitively, by bidding a higher price $P$, a buyer decreases her expected payoff in two ways. First, a higher bid increases the cost for acquiring a piece of asset, and directly reduces the her payoff. Second, a higher price $P$ also alleviates the bank run risks, making fewer banks sell for liquidity reasons. As a result, the buyer faces a pool of assets with deteriorating qualities where more banks are selling assets because of fundamental insolvency. This again reduces her expected payoff.

Having solved $P_e$, we can obtain the corresponding equilibrium critical cash flow $\theta_e \equiv \hat{\theta}(P_e)$ from expression (9). One can also verify $\theta_e \in (\theta^l, \theta^u)$.

$$\theta_e = \frac{(D_2 - D_1) + 2qD_1 - \theta + \sqrt{[(D_2 - D_1) + 2qD_1 - \theta]^2 + 4(D_2 - D_1)\theta}}{2}$$

(12)

The market equilibrium $\{\theta_e, P_e\}$ reflects asymmetric information on asset qualities. By offering $P_e$, an uninformed buyer makes a loss when the bank is insolvent, and a profit when the bank is only illiquid. Furthermore, as a lower $\theta$ aggravates the information asymmetry, it reduces the buyers’ willingness to pay, and makes banks more likely to be illiquid. Mathematically, we have $\theta_e$ decreasing in $\theta$.

Figure 3 illustrates the equilibrium funding liquidity risk. A bank with $\theta \in (D_2, \theta_e]$ may not fail and can fully repay its debt obligations if no bank run happens, yet it will fail because of premature asset liquidation caused by the run of its wholesale creditors.

**Proposition 2.** The baseline model has an unique equilibrium, with equilibrium asset price $P_e$ and equilibrium critical cash flow $\theta_e$ specified in (11) and (12) respectively. A bank with cash flow $\theta \in (D_2, \theta_e]$ is solvent but illiquid: it will fail because of a wholesale debt run, even though its assets can generate a cash flow greater than its liabilities $D_2$.

**Proof.** See Appendix B.1. □
3.5 Application I: bank capital and bank run risk

It is an entrenched belief that capital helps reduce bank run risk. An application of the current framework, however, shows that the relationship is more subtle. We show that once asset prices are endogenous, capital also contributes to bank runs via stressed asset prices.

We model an increase of bank capital in its most simplistic form. We assume that a bank maintains its unit portfolio size, increasing its equity from $E$ to $E + \Delta$, and at the same time decreasing its retail deposits from $F$ to $F - \Delta$. In other words, an increase in capital reduces $D_2$ to $D_2 - \Delta$ but does not affect $D_1$. We then examine how increasing bank capital affects the risk of bank runs. To measure bank run risks, we follow Morris and Shin (2009) and define the illiquidity risk as $IL \equiv \hat{\theta}(P) - D_2$, with $IL$ standing for illiquidity.\footnote{Strictly speaking, the illiquidity risk should be measured as the probability $\text{Prob}(D_2 < \theta < \hat{\theta}(P)) = \frac{\hat{\theta}(P) - D_2}{\theta - \theta}$.

We drop the denominator because it is a constant and does not affect comparative statistics.}

Under exogenous asset prices, a natural corollary of Proposition 1 is that a higher capital always reduces funding liquidity risks, because the cash flow generated by capital serves as an extra buffer against fire-sale losses. The value of wholesale debts is better protected and wholesale creditors have less incentive to run, a channel that we call “buffer effect”. Recall that $\hat{\theta}(P) = \frac{D_2 - D_1}{1 - \frac{q}{D_1}}$, we can write $IL$ explicitly as

$$IL = \frac{D_2 - D_1}{1 - \frac{q}{D_1}} - D_2.$$  

(13)

With price $P$ exogenous and not a function of $\Delta$, it is straightforward to verify that increasing bank capital unambiguously reduces illiquidity.

$$\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P - qD_1} < 0$$  

(14)

With endogenous asset prices, the situation is more complicated. Once investors rationally update their beliefs of a bank’s asset qualities, a higher capital level also contributes to bank runs.
by reducing endogenous fire-sale prices. The intuition is as follows. In terms of inferring the realization of \( \theta \), a bank run presents more negative news when it happens to a well-capitalized bank than when it happens to a poorly capitalized bank. Because a well-capitalized bank is able to sustain large losses, the fundamental of the bank must be unusually poor for a run to happen. With such pessimistic inference about \( \theta \), buyers’ willingness to pay for the bank’s asset decreases with the observed capital level. Therefore, a change in bank capital affects illiquidity not only via \( D_2 \) but also via endogenous asset price \( P_e \).

\[
\frac{\partial IL}{\partial \Delta} = \frac{\partial IL}{\partial D_2} \frac{\partial D_2}{\partial \Delta} + \frac{\partial IL}{\partial P_e} \frac{\partial P_e}{\partial \Delta}
\]

(15)

The first term captures the traditional “buffer effect” as in the case where the asset price is exogenous. Captured by the second term is a new channel that we want to emphasize: increasing capital also affects banks’ funding liquidity risk via endogenous asset price.

To see that higher capital leads to lower secondary market asset prices. One can simply take the first order derivative of the closed-form solution of \( P_e \), which gives

\[
\frac{\partial P_e}{\partial \Delta} = -\frac{1}{4} - \frac{1}{4} \frac{D_1 + D_2 + \theta}{\sqrt{(D_1 + D_2 + \theta)^2 - 8D_1 \theta}} < 0.
\]

Increasing capital decreases asset buyers’ willingness to pay for a bank’s assets on sale, which in turn makes creditors panic and bank runs more likely. And this is captured by

\[
\frac{\partial IL}{\partial P_e} > 0.
\]

Hence, capital can contribute to funding liquidity risk by reducing endogenous asset prices, a mechanism we dub “inference effect”. Comparing expression (14) with (15), it should be clear that with endogenous asset price and the “inference effect”, capital is less able to contain bank run risks as compared to the case where asset price is exogenous. Buyers’ rational beliefs limit the role of capital in containing funding liquidity risk.

The overall impact of capital on funding liquidity risk depends on the relative strength of the “buffer effect” and the “inference effect”. Using the closed form solution of \( P_e \) and \( \theta_e \), one can write the overall impact of an increase in capital explicitly.

\[
\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P_e - qD_1} + \frac{q(D_2 - D_1)D_1}{4(P_e - qD_1)^2} \left[ \frac{1}{4} + \frac{D_1 + D_2 + \theta}{4 \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1 \theta}} \right]
\]

(16)
It can be shown that in an extreme case where $\theta = 0$, $\partial IL/\partial \Delta = 0$ and increasing capital cannot reduce funding liquidity risk at all. Intuitively, a lower $\theta$ reduces the expected quality of assets on sale, and therefore reduces buyers’ willingness to pay. That is,

$$\frac{\partial}{\partial \theta} \left( \frac{\partial P_e}{\partial \Delta} \right) > 0.$$ 

Such drop of price is most pronounced when $\theta = 0$. In that case, the “inference effect” reaches its maximum and completely offsets the “buffer effect” of capital. We summarize the results in the following proposition.

**Proposition 3.** In equilibrium, higher bank capital leads to a lower fire-sale asset price. Compared to the case where the price is exogenous, capital is less able to reduce the risk of illiquidity. And in an extreme case where $\theta = 0$, higher capital does not reduce bank illiquidity at all.

**Proof.** See Appendix B.2

The result suggests that the design of prudential regulations has to take into account the responses of market participants. Compared to the situation where regulations are lax, market participants’ interpretation of the same piece of negative news can be more pessimistic under stringent regulations. When they panic according to their pessimistic beliefs, the effectiveness of stringent prudential regulations will be reduced, or even completely wiped out.

## 4 Self-fulfilling bank runs and financial contagion

In this section, we extend the baseline model to include two banks and two states. Asset buyers will be able to update their beliefs about State $s$ based on different numbers of bank runs. They perceive $s = B$ to be more likely when more bank runs are observed. In the absence of commitment power, the equilibrium prices that buyers offer must reflect their posterior beliefs, and therefore vary with the number of runs. We characterize market equilibrium with a single bank run and that with two bank runs, respectively. We show that for a given $N \in \{1, 2\}$, there exists a unique market equilibrium characterized by $\{P^*_N, \theta^*_N\}$ (section 4.1 and 4.2). We further establish that financial contagion can arise as a multiple-equilibria phenomenon, highlighting how pessimistic beliefs can drive financial instability (section 4.3). Finally, we discuss how
an asset purchase program committed by a regulator can improve financial stability over the market equilibria (section 4.4).

4.1 Market equilibrium with a single bank run

We start with characterizing the equilibrium with a single bank run. For a given asset price $P_1$ that corresponds to the single-bank-run outcome, the bank run game has a unique threshold equilibrium characterized by $\hat{\theta}(P_1)$. So asset buyers know that a bank run happens if and only if the bank’s cash flow is lower than $\hat{\theta}(P_1)$, and update their beliefs about the aggregate state according to Bayes’ rule. Recall that $\omega_1(\hat{\theta}(P_1))$ denotes buyers’ posterior belief for State $s$ when they observe a single bank run.

$$
\omega_1^B(\hat{\theta}(P_1)) \equiv \text{Prob}(s = B|N = 1) = \frac{(\hat{\theta}(P_1) - \theta_B) (\bar{\theta} - \hat{\theta}(P_1))}{(\hat{\theta}(P_1) - \theta_B) (\bar{\theta} - \hat{\theta}(P_1)) + (\hat{\theta}(P_1) - \theta_G) (\bar{\theta} - \hat{\theta}(P_1))} = \frac{(\hat{\theta}(P_1) - \theta_B)}{(\hat{\theta}(P_1) - \theta_B) + (\hat{\theta}(P_1) - \theta_G)} \\
\omega_1^G(\hat{\theta}(P_1)) \equiv \text{Prob}(s = G|N = 1) = \frac{(\hat{\theta}(P_1) - \theta_B) (\bar{\theta} - \hat{\theta}(P_1))}{(\hat{\theta}(P_1) - \theta_B) (\bar{\theta} - \hat{\theta}(P_1)) + (\hat{\theta}(P_1) - \theta_G) (\bar{\theta} - \hat{\theta}(P_1))} = \frac{(\hat{\theta}(P_1) - \theta_B)}{(\hat{\theta}(P_1) - \theta_B) + (\hat{\theta}(P_1) - \theta_G)}$$

When the competitive asset market is in a rational expectations equilibrium, based on their posterior beliefs, asset buyers should perceive themselves breaking even by purchasing bank assets for price $P_1$. Their ex-post zero-profit condition (8) can now write as the following.

$$P_1 = \mathbb{E} \left[ \theta \mid \theta < \hat{\theta}(P_1), N = 1 \right] = \omega_1^B(\hat{\theta}(P_1)) \frac{\theta_B + \hat{\theta}(P_1)}{2} + \omega_1^G(\hat{\theta}(P_1)) \frac{\theta_G + \hat{\theta}(P_1)}{2} \quad (17)$$

A candidate equilibrium price $P_1^*$ should be a fixed point to function $\mathbb{E} \left[ \theta \mid \theta < \hat{\theta}(P_1), N = 1 \right]$. With $\theta_1^* \equiv \hat{\theta}(P_1^*)$, we can re-write the zero-profit condition (17) as a function of $\theta_1^*$.\footnote{Here we have used the fact that $\theta_1^* \equiv \hat{\theta}(P_1^*) = \frac{|D_2-D_1|}{\bar{\theta} - \theta_1^*}$, so that $P_1^* = \frac{\hat{\theta}(P_1^*)}{\bar{\theta} - \theta_1^*}$.

$$F_1(\theta_1^*) \equiv \omega_1^B(\theta_1^*) \frac{\theta_B + \theta_1^*}{2} + \omega_1^G(\theta_1^*) \frac{\theta_G + \theta_1^*}{2} - \frac{qD_1\theta_1^*}{\theta_1^* - (D_2 - D_1)} = 0 \quad (18)$$

The equation simply states that asset buyers’ net payoffs should be zero in expectation. And finding a fixed point $P_1^*$ is equivalent to finding a solution for equation (18).
For $P_1^*$ to be an equilibrium, an asset buyer must not profit by unilaterally rising her bid above $P_1^*$. In other words, function $F_1$ should not increase in $P_1$ (or equivalently, not decrease in $\theta_1$). Such monotonicity always holds, and intuition is as follows. First of all, as discussed in section 3.3, increasing price rises the cost for acquiring bank assets and leads to an deteriorating quality in the asset pool. Second, when $P_1$ increases, the risk of bank run is mitigated, and a bank selling its assets is more likely to be fundamentally insolvent rather than facing a pure liquidity problem. For a given number of bank runs observed, this suggests that $s = B$ is more likely, i.e., $\partial \omega^B_1(\hat{\theta}(P_1)) / \partial P_1 > 0$. This further reduces the buyer’s expected payoff.

**Lemma 2.** $F_1(\theta)$ monotonically increases in $\theta_1$, meaning that given a single bank run observed, a buyer’s expected payoff monotonically decreases in her bid $P_1$.

**Proof.** See Appendix B.3.

With extra complications introduced by the posterior beliefs on $s$, we can no longer obtain closed-form solutions for $P_1^*$ and $\theta_1^*$. Instead, we prove that there exists a $\theta_1^* \in (\theta^L, \theta^U(P_1^*))$ that satisfies equation (18), and a corresponding $P_1^* \in (D_1, D_2)$ that satisfies equation (17). The proof is based on the continuity of $F_1(\theta_1)$. In particular, we show that $F_1(\theta_1)$ is negative at $\theta^L$ and positive at $\theta^U(P_1)$. Furthermore, given the monotonicity of $F_1(\theta)$, once an equilibrium exists, it is also unique. As a result, the market equilibrium with one bank run can be characterized by a unique pair $\{P_1^*, \theta_1^*\}$. The result is summarized in the proposition below.

**Proposition 4.** There exist a unique equilibrium critical cash flow $\theta_1^* \in (\theta^L, \theta^U(P_1^*))$ and a corresponding unique equilibrium asset price $P_1^* \in (D_1, D_2)$ associated with one bank run. A bank with cash flow $\theta \in (D_2, \theta_1^*]$ is solvent but illiquid.

**Proof.** See Appendix B.4.

### 4.2 Market equilibrium with two bank runs

Following the same approach as in the last section, we now characterize the equilibrium with two bank runs. For a given asset price $P_2$ that corresponds to a two-bank-run outcome, a bank will fail because of a run if and only if its cash flow $\theta < \hat{\theta}(P_2)$. Again, we formulate asset
buyers’ posterior beliefs about State $s$ according to Bayes’ rule.

$$\omega_2^B (\hat{P}_2) \equiv \text{Prob}(s = B|N = 2) = \frac{(\hat{P}_2 - \theta_B)^2}{(\hat{P}_2 - \theta_B)^2 + (\hat{P}_2 - \theta_G)^2}$$

$$\omega_2^G (\hat{P}_2) \equiv \text{Prob}(s = G|N = 2) = \frac{(\hat{P}_2 - \theta_G)^2}{(\hat{P}_2 - \theta_B)^2 + (\hat{P}_2 - \theta_G)^2}.$$  

Based on the posterior beliefs, the asset buyers’ break-even condition can be written as follows.

$$P^*_2 = E[\theta| \theta < \hat{P}_2, N = 2] = \omega_2^B (\hat{P}_2) \frac{\theta_B + \hat{P}_2}{2} + \omega_2^G (\hat{P}_2) \frac{\theta_G + \hat{P}_2}{2}$$  \hspace{1cm} (19)

And the equilibrium threshold $\theta^*_2 \equiv \hat{P}(P^*_2)$ makes the following equation $F_2(\theta^*_2) = 0$.

$$F_2(\theta^*_2) \equiv \omega_2^B(\theta^*_2) \frac{\theta_B + \theta^*_2}{2} + \omega_2^G(\theta^*_2) \frac{\theta_G + \theta^*_2}{2} - \frac{qD_1\theta^*_2}{\theta^*_2 - (D_2 - D_1)} = 0$$  \hspace{1cm} (20)

Lemma 3 shows that buyers’ expected payoff monotonically decreases in $P_2$, so that they have no profitable deviation. Thus, any solution to equation (20) is indeed a market equilibrium.

**Lemma 3.** $F_2(\theta)$ monotonically increases in $\theta$, meaning that given two bank runs observed, a buyer’s expected payoff monotonically decreases in her bid $P$.

**Proof.** See Appendix B.5. \hfill \Box

To prove the existence of and uniqueness of the equilibrium, we again use the monotonicity and continuity of function $F_2(\theta_2)$. We show that $F_2(\theta_2)$ is negative at $\theta^L$ and positive at $\theta^U$, so that the market equilibrium with one bank run can be characterized by a unique pair $\{P^*_2, \theta^*_2\}$. The result is summarized in Proposition 5.

**Proposition 5.** There exist a unique equilibrium critical cash flow $\theta^*_2 \in (\theta^L, \theta^U(P^*_2))$ and a corresponding unique equilibrium asset price $P^*_2 \in (D_1, D_2)$ associated with two bank runs. A bank with cash flow $\theta \in (D_2, \theta^*_2)$ is solvent but illiquid.

**Proof.** See Appendix B.6. \hfill \Box

### 4.3 Financial contagion and multiple equilibria

$\theta^*_2 > \theta^*_1$ would imply potential contagion. In particular, when a bank’s cash flow lies between $\theta^*_1$ and $\theta^*_2$, the bank will face no run if the other bank does not face a run, and will fail
in a wholesale run if the other bank does. We prove with Lemma 4 that $\theta^*_2 > \theta^*_1$ is indeed the case. Intuitively, the asset buyers form more pessimistic beliefs about State $s$ when having observed more bank runs. Their willingness to pay for banks’ assets decreases as banks’ expected asset qualities are lower in State $B$. This in turn reduces equilibrium asset price pushes up the equilibrium critical cash flow that a bank has to meet to survive a run.

**Lemma 4.** When more runs are observed, the equilibrium market asset price is lower $P^*_2 < P^*_1$ and the risk of bank runs is higher $\theta^*_2 > \theta^*_1$.

**Proof.** See Appendix B.7. □

Financial contagion emerge as a multiple-equilibrium phenomenon in the current model. In fact, when a bank’s cash flow $\theta \in (\theta^*_1, \theta^*_2)$ and the other bank’s cash flow $\theta < \theta^*_2$, the equilibrium number of bank runs depends on creditors’ beliefs about each others’ strategies. As only two threshold strategies can be be rationalized as part of a market equilibrium, i.e., an optimistic threshold strategy, ‘to run if and only if $x < \theta^*_1$’, and a pessimistic threshold strategy, ‘to run if and only if $x < \theta^*_2$’, we can focus on those two threshold strategies only. We show that financial contagion can happen purely because of creditors’ pessimistic beliefs.

For the ease of exposition, we label the two banks as Bank $i$ and $j$, and discuss the following two cases respectively. (1) Bank $i$ has a cash flow $\theta \in (\theta^*_1, \theta^*_2)$ and Bank $j$ has a cash flow $\theta < \theta^*_1$.\textsuperscript{16} And (2) Bank $i$ and $j$ both have cash flows between $\theta^*_1$ and $\theta^*_2$.

In the first case, the equilibrium number of bank runs can be either 1 or 2, depending on creditors’ belief about each others’ strategies. With a cash flow $\theta < \theta^*_1$, Bank $j$ will fail in a run whether creditors follow the optimistic or pessimistic strategy. Therefore, there will be at least one bank run in the economy. Whether Bank $i$ will have a run, however, depends on creditors’ beliefs. If creditors believe that a positive mass among them follow the pessimistic strategy, they will expect a run on Bank $i$ and an asset price $P^*_2$, so that it is optimal to join the run. As a result, that all creditors withdraw early from Bank $i$ can emerge as an equilibrium. On contrast, if all creditors believe that none of them follow the pessimistic strategy, they would expect the asset price to be $P^*_1$, and only Bank $j$ to fail, which justifies their optimistic belief/strategy in the first place.

In the second case, the equilibrium number of bank runs can be either 0 or 2, depending again on creditors’ beliefs. If all creditors believe that none of them follow the pessimistic

\textsuperscript{16}The symmetric case where Bank $i$ has a cash flow $\theta < \theta^*_1$, and Bank $j$ has $\theta \in (\theta^*_1, \theta^*_2)$ can be analyzed with the same reasoning.
strategy, no run will happen, because both banks’ cash flows are higher than $\theta^*_1$. Therefore, $N = 0$ can be an equilibrium. On contrast, if a creditor believes that a positive mass among them follow the pessimistic strategy, he will expect two bank runs and assets sold for price $P^*_2$, so that it is optimal for him to join the run. Therefore, $N = 0$ can emerge as an equilibrium. The creditor’s belief must be that a positive mass of creditors will run both banks. This is because if the pessimistic creditors are present in one bank, then those creditors’ strategy cannot be rationalized. Therefore, $N = 1$ cannot be an equilibrium.

In sum, multiple equilibria can emerge when a bank’s cash flow in $[\theta^*_1, \theta^*_2]$ and the other bank’s cash flow below $\theta^*_2$. The contagion is self-fulfilling and can be fuelled completely by creditors’ beliefs. In Figure 4, we plot the possible equilibrium outcomes for different combinations of bank cash flows, and summarize the results in Proposition 6.

**Proposition 6.** When one bank’s cash flow belongs to $[\theta^*_1, \theta^*_2]$ and the other bank’s cash flow is lower than $\theta^*_2$, multiple market equilibria exist, and financial contagion can happen because of creditors’ pessimistic beliefs.

### 4.4 Application II: Asset purchase program

We show in this section that a regulator with commitment power can promote financial stability even if he is not better informed than the asset buyers. The welfare-improving policy
intervention that we propose resembles asset purchase programs such as Term Asset-Backed Securities Loan Facility (TALF).

We consider the following policy intervention: the regulator makes a promise to purchase bank assets at a unified price $P_A$ in case any bank run happens. In particular, the unified price $P_A$ does not depend on the number of bank runs in the economy. The regulator is assumed to have full commitment power and will not revoke his offer after having observed the actual number of bank runs. Under the policy intervention, the model has the following revised timeline.

Figure 5: Timing of the game: with announced price

<table>
<thead>
<tr>
<th>t = 0</th>
<th>t = 0.5</th>
<th>t = 1</th>
<th>t = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Banks are established, with their portfolios and liability structures as given.</td>
<td>The regulator makes a promise to buy assets at a unified price $P_A$, in case any bank run happens.</td>
<td>1. $s$ and $\theta$ are realized. 2. Asset buyers post a price scheme. 3. For each bank that they lend to, creditors receive private noisy signals about the bank’s cash flow $\theta$, and decide to run or not. 4. Assets are sold to the party that offers the highest price.</td>
<td>1. Returns are realized. 2. Remaining obligations are settled</td>
</tr>
</tbody>
</table>

The regulator is risk-neutral, and subject to an ex-ante budget constraint: he should not make any loss in expectation. To maximise social welfare, he will choose an optimal price $P^*_A$ so that he only breaks even. This is because any higher price that leads to a positive expected profit will come at a cost of letting more solvent banks fail in runs.

The regulator is different from the ordinary asset buyers in the market because he holds full commitment power. In particular, he does not require to break even for each observed number of bank runs, but only to break even ex ante. The commitment power allows the regulator to disregard new information such as the number of bank runs, and can therefore avoid the vicious cycle fuelled by pessimistic belief updating.

We now derive the ex-ante break-even price $P^*_A$. As a first step, we solve for price $P^*_A$, under the assumption that banks will sell their assets to the regulator instead of to the asset buyers in the secondary market. When wholesale creditors expect bank assets to be sold at price $P_A$, we know from Section 3.3 that the critical cash flow of the bank run game is $\hat{\theta}(P_A)$. So the regulator understands that only those assets with $\theta < \hat{\theta}(P_A)$ will be on sale, with $\hat{\theta}(P_A)$ again
defined by expression (7).

\[ \hat{\theta}(P_A) = \frac{D_2 - D_1}{1 - qD_1/P_A} \]  

(21)

As the regulator commits to price \( P_A \) before observing any number of bank runs, he holds the prior belief \( \text{Prob}(s = G) = \text{Prob}(s = B) = 1/2 \). From this ex-ante perspective, the regulator’s break-even condition can be written as follows.

\[ P^*_A = \frac{1}{2} \frac{\theta_B + \hat{\theta}(P^*_A)}{2} + \frac{1}{2} \frac{\theta_G + \hat{\theta}(P^*_A)}{2} \]  

(22)

Using expression (21), we can rewrite equation the ex-ante break-even condition into a quadratic function of \( P^*_A \), which has the following root between \( D_1 \) and \( D_2 \).

\[ P^*_A = \frac{[2(D_2 - D_1) + 4qD_1 + (\theta_B + \theta_G)] + \sqrt{[2(D_2 - D_1) + 4qD_1 + (\theta_B + \theta_G)]^2 - 16qD_1(\theta_B + \theta_G)}}{8} \]  

(23)

Having obtained \( P^*_A \), we can derive the corresponding \( \hat{\theta}(P^*_A) \) using equation (21). Following the same procedure in the proof of Proposition 4, we can prove that \( \hat{\theta}(P^*_A) \in (\theta_L, \theta_U(P^*_A)) \), so that the policy intervention cannot completely eliminate inefficient bank runs.

**Lemma 5.** Suppose that facing runs, banks can sell their assets at a unified price committed by the regulator. The regulator can break even ex ante by offering price \( P^*_A \) as in (23). And the bank run game has a threshold equilibrium where a run happens if and only if \( \theta < \hat{\theta}(P^*_A) \).

**Proof.** See Appendix B.8. □

Now one can verify that \( P^*_A \) is higher than what market offers (i.e., \( P^*_A > P^*_1 > P^*_2 \)), so that banks will indeed sell their assets to the regulator. As \( \hat{\theta}(P) \) decreases with \( P \), the policy intervention improves financial stability as compared to the market equilibria. In particular, the asset purchase committed by the regulator reduces (though does not eliminate) the risk of bank runs, and completely rules out financial contagion. The result is summarised in Proposition 7.

**Proposition 7.** The regulator’s ex-ante break-even price \( P^*_A \) is higher than the prices in market equilibria, so that banks will sell their assets to the regulator when they face runs. With \( P^*_A > P^*_1 > P^*_2 \) and \( \hat{\theta}(P^*_A) < \hat{\theta}(P^*_1) < \hat{\theta}(P^*_2) \), the regulator can reduce bank run risks and eliminate financial contagion.

**Proof.** See Appendix B.9. □
With his commitment power, the regulator can disregard the outcome of bank run games and stick to a unified asset price. The commitment power allows the regulator to avoid the viscous cycle between bank runs and fire sales that is fueled by pessimistic beliefs in market. As the regulator only needs to break even ex ante given his prior belief about State \( s \), he can use the profit from State \( G \) to compensate the loss in State \( B \).

The ordinary buyers in market, on the other hand, are unable to do so. Without commitment power, they must not make expected loss given any realized number of bank runs. In other words, they are constrained by ex-post break-even conditions. In fact, if an asset buyer offers the same price \( P_A^* \), she will revoke the offer when a bank run actually happens, because in that case she will form a posterior belief that \( s = B \) is more likely and will no longer consider herself breaking even by purchasing bank assets at \( P_A^* \). To break even from this ex-post perspective, the asset buyer has to lower her offered price, so as to decrease the loss from purchasing assets with \( \theta \in [\theta_s, P_N^*] \), and to increase the profit from purchasing assets with \( \theta \in [P_N^*, \hat{\theta}(P_N^*)] \). The lack of commitment power therefore leads to lower asset prices, which in turn result in more bank runs, and justify the pessimistic beliefs in the first place.

5 More policy discussion

It should be noticed that the model is sufficiently rich for other policy analyses. We present here one more application on regulatory transparency.\(^{17}\)

From the discussion in the above sections, the lack of information on aggregate state contributes to the financial instability. It seems that the promotion of information transparency on aggregate risk can serve as an remedy to restore financial safety. In fact, this seemingly natural solution can lead to even greater market distress. To shed some light on this discussion, we consider a situation where a regulator has superior information about aggregate state \( s \) and can credibly convey the information to the market. While it could reduce the illiquidity caused by the buyers’ pessimistic belief updating in the good state, regulatory transparency exacerbates the situation when the state is bad. Thus, the regulator faces a tradeoff when making the ex ante disclosure decision.

\(^{17}\)In reality, the examples of regulatory transparency are such as the establishment of an early warning system, the release of stress testing parameters or the announcement of the size of assistance program.
5.1 Trade-offs for regulatory transparency

We assume that legislation allows the regulator to perfectly commit to disclose the true information. To concentrate on the effects of disclosure, we consider a simplistic case where the regulator observes the aggregate state \( s \) perfectly. Once the regulator commits to disclose information, the information concerning the aggregate state will be released before market trading. The information set of creditors and asset buyers changes correspondingly. In contrast to the belief updating about state upon observing the number of bank runs, now buyers know with certainty the aggregate state. Therefore the price conditions on the true state that the regulator discloses.

For brevity, we omit the derivation of the rational expectations equilibrium. Let \( \theta^G_e, \theta^B_e, P^G_e \) and \( \theta^B_e, P^B_e \) denote the equilibrium critical cash flow and asset price in the good state and in the bad state respectively. The following Lemma 5 shows the effect of disclosing aggregate state on the banks’ illiquidity.

**Lemma 6.** Under regulatory disclosure there exist a unique critical cash flows \( \theta^s_e (s = G, B) \) corresponding to the true state \( s \) \((s = G, B)\). In the true state \( s \), banks with cash flows \( \theta \in (D_2, \theta^s_e] \) are solvent but illiquid. The regulatory announcements eliminates the multiple equilibria caused by the asset buyers’ beliefs about the aggregate state.

*Proof.* See Appendix B.10. \( \square \)

The asset buyers now know the aggregate state after hearing the announcement. There is no needs to form beliefs about the realization of \( s \) based on the observation of bank runs. The regulatory disclosure eliminates different inferences as a source of multiple equilibria: Instead of two rational expectations equilibria depending on buyers’ the beliefs, there is a single equilibrium depending on the announced (realized) state.

Intuitively, \( \theta^G_e < \theta^*_1 \) and \( \theta^B_e > \theta^*_2 \). Even if observing two bank runs, buyers cannot be certain that \( s = B \). But as as long as the regulatory disclosure is accurate, buyers will lower their valuation of assets further if \( s = B \) is communicated. Similarly, making a favorable disclosure will save certain banks from illiquidity, as market participants are reassured. We now show this rigorously.

**Proposition 8.** \( \theta^G_e < \theta^*_1 \) and \( \theta^B_e > \theta^*_2 \). The regulatory announcement reduces illiquidity if \( s = G \) but increases it if \( s = B \).
Proof. See Appendix B.11.

Graphically, we have Figure 7.

Figure 6: Regulatory Transparency

The disclosed information, when favorable, reduces adverse selection: banks with \( \theta \in (\theta_G^e, \theta_1^e) \) are saved from bank runs. However, acknowledge a crisis in the bad state will exacerbate liquidity problems: a solvent bank is more likely to suffer from illiquidity when market participants are more aware that the whole economy is in the bad state. In particular, banks with \( \theta \in (\theta_2^u, \theta^B) \) will for sure be confronted with runs. Therefore, in determining whether to commit to disclose information, the regulator faces a trade-off: if the state is good, it saves banks from illiquidity; if the state is bad, transparency will create even more panic and runs by pushing asset prices further down. Intuitively, when social cost of bank failure in the crisis state, i.e., \( s = B \) is sufficiently large, it is suboptimal for the regulator to disclose the information to market.

5.2 Regulatory transparency vs. Asset purchase

Now we run a horse racing between different policy interventions, examining whether regulatory transparency can outperform an asset purchase program as modelled in section 4.4. We show that an asset purchase program, which does not require superior information on the aggregate state, can actually achieve a higher level of financial stability.

Indeed, asset purchase program and regulatory transparency are mutually exclusive. Once credibly disclosing the true state to be \( G \), the equilibrium market price will be \( P_G^e \) higher than the announced price \( P_A^e \) derived in section 4.4. The reason is again, the market participants acknowledge the state is indeed good, the creditors are reassured and the asset buyers’ willingness to pay is highest. Suppose asset purchase program and regulatory disclosure coexist, it will be impossible for the regulator to purchase any assets in the good state unless his announced price is even higher than \( P_G^e \). However, it is never credible for the regulator to commit to purchase assets at such a price as he makes losses even from an ex ante perspective.
We now conduct a cost and benefit analysis to evaluate the regulator’s optimal policy choice between an asset purchase program and regulatory transparency. For simplicity, we concentrate on the social cost of bank failure. We denote this social cost to be \( C \) and assume it is independent of the number of bank runs (failures) and the residual cash flows. The regulator’s objective, then is to choose the policy intervention, which minimizes the expected social cost of bank failure. We denote \( SC^{AP} \) and \( SC^{RT} \) as the expected social costs when implementing the asset purchase program and the regulatory transparency, respectively.

From section 4.4, \( SC^{AP} \) can be formulated as

\[
SC^{AP} = \frac{1}{2} \cdot \frac{\hat{\theta}(P^A) - \theta_B}{\theta - \theta_B} C + \frac{1}{2} \cdot \frac{\hat{\theta}(P^A) - \theta_G}{\theta - \theta_G} C.
\] (24)

Recall that from section 4.4, \( \hat{\theta}(P^s_A) \) is the critical cash flow when the regulator makes the ex ante break even announcement \( P^s_A \).

Note that \( \theta^s_e = \hat{\theta}(P^s_e) \ (s = G, B) \). On the other hand, the regulator’s objective when implementing the regulatory transparency policy is

\[
SC^{RT} = \frac{1}{2} \cdot \frac{\hat{\theta}(P^B) - \theta_B}{\theta - \theta_B} C + \frac{1}{2} \cdot \frac{\hat{\theta}(P^G) - \theta_G}{\theta - \theta_G} C.
\] (25)

Proposition 9 shows that the social cost when the regulator implements asset purchase program is strictly lower.

**Corollary 1.** Social cost due to bank failure is lower when the regulator implements the asset purchase program \( SC^{AP} < SC^{RT} \). The economy is better off when the regulator suppress his superior information regarding to the true state.

**Proof.** See Appendix B.12. \( \square \)

Intuitively, the result depends on the critical cash flow as a function of \( P, \theta(P) \) is decreasing and convex in \( P \).

### 6 Concluding remarks

In this paper, we investigate the relationship between fire-sales and bank runs. We present a model where fire-sale prices and bank runs, driven by asymmetric information and buyers’
belief updating, are endogenously determined in rational expectations equilibrium. Furthermore, we extend the model to incorporate contagion when there is a common risk exposure. We draw several results from our analysis. First, fire-sales and bank runs are self-fulfilling and mutually reinforcing: when creditors anticipate low prices for a bank’s asset sales, a run will be triggered, which generates fire-sales and the corresponding collapse in prices, thus fully justifying creditors’ strategies. Second, as a bank fails, asset buyers lower their expectation of common risk factor and perceive banks’ asset to be less valuable: the declining asset price will precipitate runs at all other banks.

Based on the model, we draw policy implications regarding capital and regulatory transparency. We show that high capital overall makes the banking industry more resilient against systemic crises. Also, complementary to conventional wisdom, capital can also have side effects on both illiquidity and contagion because buyers’ inference via endogenous fire sale prices. A run presents more negative news, both for idiosyncratic and common risk factors, when it happens to a well-capitalized bank. Asset buyers’ perceived asset quality will deteriorate further compared to the case where runs are on poorly capitalized banks. As buyers’ willingness to pay drops more sharply, it is more likely that creditors panic such that funding liquidity dries up and contagion starts. We also show that regulatory transparency is a double-edged sword: On the one hand, it eliminates the multiple equilibria due to the buyers’ beliefs about the aggregate state. On the other hand, it saves illiquid banks when the disclosure is favorable. However, it amplifies illiquidity and contagion problem when the disclosure deteriorates market beliefs. When systemic crisis is more costly than individual bank failures, the desirability of revealing aggregate risk is open to question.

References


Appendix A  Bank run game for $D_1 < P_N < D_2$

In this section, we solve the creditors’ bank run game for a given secondary market price $P_N$ belongs to the interval $(D_1, D_2)$.

Appendix A.1  Lower and upper dominance regions

Following the standard procedure of global games, we start with the lower dominance region denoting as $[\theta_{L}, \theta_{U}]$. Suppose all other creditors stay until $t = 2$ when the bank’s cash flow realizes in this region, then the fraction of creditors who withdraw is $L = 0$. Under this case, there is no bank run. In this circumstance, a creditor $i$ still withdraws at $t = 1$ if and only if the inequality (5) holds for $L = 0$, that is $\theta \leq F + (1 - E - F)r_D = D_2$. The bank’s fundamental is so poor that the bank still fails at $t = 2$ even if there is no premature liquidation of its assets. A creditor $i$ who waits will get zero because of the bankruptcy. Instead, he will get $q_rD$ if withdrawing early. Thus, we define $\theta_{L} = D_2$. In our analysis, the support of noise $\varepsilon$ is taken to be arbitrarily close to zero, so creditors are sure when the bank’s cash flow realizes in $[\theta, D_2)$. Thus, a creditor’s dominant strategy is to withdraw at $t = 1$ to get $q_rD$ in this circumstance.

Second, we choose $\theta_{U}(P_N) = \frac{F}{1 - D_1/P_N}$ given the asset price $P_N$. Then the upper dominance region is $[\theta_{U}(P_N), \theta]$. Note that we can always have

$$\bar{\theta} > \frac{F}{1 - D_1/P_N}$$

by assuming $\bar{\theta}$ is sufficiently large to keep the existence of the upper dominance region. Now suppose all other creditors withdraw early when the bank’s fundamental realizes in the upper dominance region, then $L = 1$. Under this case, a successful bank run always occurs irrespective of the bank’s cash flow. Yet, the bank still survives at $t = 2$ if the inequality (5) does not hold, $(1 - D_1/P_N)\theta > F$. In other words, the bank always survives if its realized cash flow is sufficiently large $\theta > \frac{F}{1 - D_1/P_N}$. Again, the creditors’ signals are arbitrarily accurate, they are sure when the bank’s cash flow realizes in $(\theta_{U}(P_N), \bar{\theta})$. A creditor’s dominant strategy is to stay until $t = 2$ to avoid the penalty for early withdrawal ($r_D > q_rD$).
Appendix A.2  Beliefs of creditors outside the dominance regions

In this subsection, we characterize creditors’ beliefs when the bank’s cash flow realizes in the intermediate region \((\theta^L, \theta^U(P_N))\). Now creditors’ actions depend on their beliefs about the actions of other creditors. The signals regarding to the bank’s realized cash flow form their beliefs.

To proceed, we first determine the fraction of creditors who withdraw at \(t = 1\) as a function of a bank’s realized cash flow and the threshold. Formally, when a bank’s cash flow \(\theta\) realizes in the region \((\theta^L, \theta^U(P_N))\), a creditor \(i\) receives a signal \(x_i = \theta + \epsilon_i\), with \(\epsilon_i \sim U(-\epsilon, \epsilon)\) as the noise about the realized fundamental. We suppose each creditor acts according to a threshold strategy and set the threshold signal as \(\hat{x}\), i.e., a creditor \(i\) withdraws at \(t = 1\) if \(x_i < \hat{x}\), stays until \(t = 2\) if \(x_i > \hat{x}\). The fraction of creditors who withdraw at \(t = 1\) should be a function of the realized cash flow \(\theta\) and the threshold of signals \(\hat{x}\), that is \(L(\theta, \hat{x})\). This is because the decision to withdraw or stay depends on both the realization of the cash flow and the strategy of other players. To achieve model tractability, we follow the classic approach in global games by assuming that the creditors’ signal about the realized cash flow is sufficiently accurate. The noise \(\epsilon_i\) is distributed on an arbitrarily small interval, \(\epsilon \to 0\). As a result, we can consider the threshold of signal \(\hat{x}\) approximately to be a threshold of bank’s cash flow \(\hat{\theta}\), as \(\hat{x}\) and \(\hat{\theta}\) are arbitrarily close. Then a representative creditor \(i\) withdraws at \(t = 1\) if \(x_i < \hat{\theta}\), stays till \(t = 2\) if \(x_i > \hat{\theta}\) and the fraction of early withdrawals is \(L(\theta, \hat{\theta})\).

Our second step is to determine the functional form of \(L(\theta, \hat{\theta})\). For a realized \(\theta\), we have three cases: (i) When \(\theta + \epsilon < \hat{\theta}\), even the highest possible signal is below the threshold \(\hat{\theta}\). According to the definition of threshold strategy, all creditors withdraw at \(t = 1\), and \(L(\theta, \hat{\theta}) = 1\). (ii) When \(\theta - \epsilon > \hat{\theta}\), even the lowest possible signal exceeds the threshold \(\hat{\theta}\). Then all creditors stay till \(t = 2\). (iii) When \(\theta\) falls into the intermediate range \([\hat{\theta} - \epsilon, \hat{\theta} + \epsilon]\), the fraction of creditors who withdraw at \(t = 1\) is determined as

\[
L(\theta, \hat{\theta}) = \text{Prob}(x_i < \hat{\theta} | \theta) = \text{Prob}(\epsilon_i < \hat{\theta} - \theta | \theta) = \frac{\hat{\theta} - \theta - (-\epsilon)}{2\epsilon} = \frac{\hat{\theta} - \theta + \epsilon}{2\epsilon}. \quad (A.26)
\]

A creditor who receives a signal \(x_i\) holds a posterior belief that the fundamental follows a uniform distribution on \([x_i - \epsilon, x_i + \epsilon]\) because the noise \(\epsilon_i\) is uniformly distributed on \([-\epsilon, \epsilon]\). As the proportion of creditors who withdraw is a function of the fundamental, each creditor forms a posterior belief about the proportion.
The third step is to derive those posterior beliefs. To begin with, we show that the distribution is uniform on \([0, 1]\) for the marginal creditor who happens to observe \(s_i = \hat{\theta}\). Indeed, we have

\[
Prob \left( L(\theta, \hat{\theta}) \leq \hat{L} \mid x_i = \hat{\theta} \right) = Prob \left( \frac{\hat{\theta} - \theta + \epsilon}{2\epsilon} \leq \hat{L} \mid x_i = \hat{\theta} \right) = Prob \left( \theta \geq \hat{\theta} + \epsilon - 2\epsilon \hat{L} \mid x_i = \hat{\theta} \right).
\]

On the other hand, we know that, conditional on \(x_i = \hat{\theta}\), the marginal creditor has a posterior belief that \(\theta\) is uniformly distributed on \([\hat{\theta} - \epsilon, \hat{\theta} + \epsilon]\), which implies \(Prob \left( L(\theta, \hat{\theta}) \leq \hat{L} \mid x_i = \hat{\theta} \right) = \hat{L}\). Therefore, the marginal creditor holds a posterior belief that the fraction of creditors who withdraw at \(t = 1\) forms a uniform distribution on \([0, 1]\), that is \(L(\theta, \hat{\theta}) \mid x_i = \hat{\theta}) \sim U(0, 1)\).

We then move onto the slightly more complicated cases for the non marginal creditor, \(x_i \neq \hat{\theta}\). Without loss of generality, we start with the case \(x_i > \hat{\theta}\). Remember that a creditor who receives a signal \(x_i\) holds a posterior belief that the fundamental follows a uniform distribution on \([x_i - \epsilon, x_i + \epsilon]\). Given \(x_i > \hat{\theta}\), the upper bound of the support is greater than \(\hat{\theta} + \epsilon\). And we know that when \(\theta > \hat{\theta} + \epsilon\), all creditors stay and \(L = 0\). In fact, we can divide the support of \(\theta\) into two sections: \([x_i - \epsilon, \hat{\theta} + \epsilon]\) and \([\hat{\theta} + \epsilon, x_i + \epsilon]\). As we have discussed, the second section corresponds to a posterior belief \(L(\theta, \hat{\theta}) \mid x_i = \hat{\theta}) = 0\). Therefore in the eyes of a creditor \(i\) who receives \(x_i > \hat{\theta}\), there will be a positive probability mass on \(L = 0\). On the other hand, we can show that the posterior belief of \(\theta\) continues to be a uniform distribution on \([x_i - \epsilon, \hat{\theta} + \epsilon] \subset [\hat{\theta} - \epsilon, \hat{\theta} + \epsilon]\).

Since \(\theta\) is again within the intermediate range \([\hat{\theta} - \epsilon, \hat{\theta} + \epsilon]\), the expression of \(L(\theta, \hat{\theta})\) will follow expression (A.26), and we can derive the posterior belief on \(L\) as follows.

\[
Prob \left( L(\theta, \hat{\theta}) \leq \hat{L} \mid x_i \right) = \frac{L \mid x_i}{L \mid x_i = \hat{\theta}} = \frac{L \mid x_i}{L \mid x_i = \hat{\theta}} = \frac{L \mid x_i}{L \mid x_i = \hat{\theta}} + \frac{L \mid x_i}{L \mid x_i = \hat{\theta}}.
\]

Because the player perceives a uniform distribution of \(\theta\) on \([x_i - \epsilon, \hat{\theta} + \epsilon]\), the probability above can be calculated as \(\frac{L}{1 - (x_i - \hat{\theta})/2\epsilon}\), and this is a uniform distribution on \([0, 1 - \frac{x_i - \hat{\theta}}{2\epsilon}]\). Notice that the density function on this interval is 1, thus the probability uniformly allocated on this interval is \(1 - \frac{x_i - \hat{\theta}}{2\epsilon}\), and the probability mass at \(L(\theta, \hat{\theta}) = 0\) is \(Prob \left( L(\theta, \hat{\theta}) = 0 \right) = \frac{x_i - \hat{\theta}}{2\epsilon}\). A creditor who observes \(x_i > \hat{\theta}\) holds a more optimistic belief that a smaller proportion of creditors will withdraw (reflected by the positive probability mass on \(L = 0\) where no one withdraws). As the marginal creditor who observes \(x_i = \hat{\theta}\) is indifferent between withdrawing or not, the player who observes \(x_i > \hat{\theta}\) will prefer to stay. Moreover, the higher the signal \(s_i\) received, the more optimistic belief a creditor \(i\) holds (\(Prob \left( L(\theta, \hat{\theta}) = 0 \right) = \frac{x_i - \hat{\theta}}{2\epsilon}\) increases in \(x_i\).
The case \( x_i < \hat{\theta} \) follows exactly the same procedure. We can show that from the perspective of a creditor who observes \( x_i < \hat{\theta} \), \( L \) has a mixed distribution: it is uniformly distributed on \([\frac{\hat{\theta} - x_i}{2\epsilon}, 1]\) with density function 1, and has with a positive probability mass at \( L(\theta, \hat{\theta}) = 1 \). The probability mass at \( L = 1 \) is \( \text{Prob} \left( L(\theta, \hat{\theta}) = 1 \right) = \frac{\hat{\theta} - x_i}{2\epsilon} \), where creditor \( i \) believes every one withdraws. Thus, a creditor who observes \( x_i < \hat{\theta} \) will be more pessimistic and prefer to withdraw. Moreover, the lower the signal \( s_i \) received, more pessimistic belief a creditor \( i \) holds \((\text{Prob} \left( L(\theta, \hat{\theta}) = 1 \right) = \frac{\hat{\theta} - x_i}{2\epsilon} \) increases in \( x_i \)).

Appendix A.3 Threshold Equilibrium

The previous subsections show that upper and lower dominance regions are existent and any creditor whose signal is higher (lower) than \( \hat{\theta}(P_N) \) is more prone to stay (withdraw). Now we formally derive the value of this critical cash flow by the indifference condition of the marginal creditor. Remember that the marginal creditor, observing exactly \( \hat{\theta} \), is indifferent between stay and withdraw. We have derived that his belief is \( L \sim U(0, 1) \) and formulated the difference \( DW(L) \) in the section 3.2. The creditor’s indifference condition then can be expressed as

\[
\int_{0}^{1} DW(L) dL = 0,
\]

or

\[
\int_{L'}^{1} qr_d dL - \int_{0}^{L'} (1 - q) r_d dL = qr_d(1 - L') - (1 - q) r_d L' = 0.
\]

Recall the definition of \( L' \), \( L' = \frac{P_N(\theta - D_2)}{(\theta - P_N) / P_N} \). The indifference condition implies a unique critical cash flow \( \hat{\theta} \) for a given asset price \( P_N \in (D_1, D_2) \).

\[
\hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1 / P_N},
\]

For a given asset price \( P_N \in (D_1, D_2) \), a run happens to banks with \( \theta < \hat{\theta}(P_N) \). Geometrically, we present the indifference condition in Figure 3.
Appendix B  Proofs to Lemmas and Propositions

Appendix B.1  Proposition 2. Solution to the baseline model

Proof. To solve the equilibrium critical cash flow $\theta_e$, note that $?\theta?\theta$ is actually a quadratic function of $\hat{\theta}$

$$\hat{\theta}^2 - [(D_2 - D_1) + 2qD_1 - \hat{\theta}] \hat{\theta} - (D_2 - D_1)\hat{\theta} = 0.$$ 

Using the quadratic formula, we can obtain two solutions and retain the positive one

$$\theta_e = \frac{(D_2 - D_1) + 2qD_1 - \hat{\theta} + \sqrt{[(D_2 - D_1) + 2qD_1 - \hat{\theta}]^2 + 4(D_2 - D_1)\hat{\theta}}}{2}.$$ 

The equilibrium asset price $P_e$ can be obtained by solving (10) or directly from the zero profit condition $P_e = \frac{\theta_e + \theta}{2}$. We have

$$P_e = \frac{(D_2 - D_1) + 2qD_1 + \hat{\theta} + \sqrt{[(D_2 - D_1) + 2qD_1 + \hat{\theta}]^2 - 8qD_1\hat{\theta}}}{4},$$ 

Note that $[(D_2 - D_1) + 2qD_1 - \hat{\theta}]^2 + 4(D_2 - D_1)\hat{\theta} = [(D_2 - D_1) + 2qD_1 + \hat{\theta}]^2 - 8qD_1\hat{\theta}.$

To prove $\theta_e > D_2$ and $P_e \in (D_1, D_2)$, we let $q$ be sufficiently close to 1 to simplify the calculation. Note that this assumption is innocuous as $q$ is the penalty for early withdrawal, in
With the analytical form of $P_\theta$ given by

$$
\theta = \frac{(D_1 + D_2 - \theta) + \sqrt{(D_1 + D_2 - \theta)^2 + 4P_\theta}}{2}, \quad P_\theta = \frac{(D_1 + D_2 + \theta) + \sqrt{(D_1 + D_2 + \theta)^2 - 8qD_1\theta}}{4},
$$

We show that increasing capital is less able, or even has no effect in reducing a bank’s illiquidity risk when asset price is endogenous.

Proof. We show that increasing capital is less able, or even has no effect in reducing a bank’s illiquidity risk when asset price is endogenous.

From (16), we obtain

$$
\frac{\partial \Pi}{\partial \Delta} = \frac{-qD_1}{P_e - qD_1} \frac{(P_e - qD_1) \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1\theta} - (D_2 - D_1)P_e}{(P_e - qD_1) \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1\theta}}
$$

Provided that $P_e > D_1$, we have $\text{sgn} \left( \frac{\partial \Pi}{\partial \Delta} \right) = -\text{sgn} \left[ (P_e - qD_1) \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1\theta} - (D_2 - D_1)P_e \right].$

With the analytical form of $P_e$ from Appendix B.2, we have

$$
\text{sgn} \left[ (P_e - qD_1) \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1\theta} - (D_2 - D_1)P_e \right] = \text{sgn} \left[ D_1(D_2 - P_e) + (\theta - D_1)(P_e - D_1) \right]
$$

Thus, there exists an interval for $q$ such that when $q \in (1 - \epsilon, 1)$, $\frac{d}{dq} \left( \theta^U(P_e) - \theta_c \right) < 0$ if and only if $P_e < D_2$. Combining with $\theta_c = \theta^U(P_e)$ when $q = 1$, we obtain $\theta^U(P_e) > \theta_c$ when $q \in (1 - \epsilon, 1)$. That is $\theta_c \in [D_2, \theta^U(P_e)] \subset [D_2, \theta]$. To conclude, $\theta_c$ and $P_e$ derived above is the unique equilibrium critical cash flow and asset price in the baseline model. Thus, the creditors’ beliefs and asset buyers’ beliefs are consistent.

\[\square\]

**Appendix B.2  Proposition 3. Bank capital and illiquidity**

Proof. We show that increasing capital is less able, or even has no effect in reducing a bank’s illiquidity risk when asset price is endogenous.

From (16), we obtain

$$
\frac{\partial \Pi}{\partial \Delta} = \frac{-qD_1}{P_e - qD_1} \frac{(P_e - qD_1) \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1\theta} - (D_2 - D_1)P_e}{(P_e - qD_1) \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1\theta}}
$$

Provided that $P_e > D_1$, we have $\text{sgn} \left( \frac{\partial \Pi}{\partial \Delta} \right) = -\text{sgn} \left[ (P_e - qD_1) \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1\theta} - (D_2 - D_1)P_e \right].$

With the analytical form of $P_e$ from Appendix B.2, we have

$$
\text{sgn} \left[ (P_e - qD_1) \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1\theta} - (D_2 - D_1)P_e \right] = \text{sgn} \left[ D_1(D_2 - P_e) + (\theta - D_1)(P_e - D_1) \right]
$$

Thus, there exists an interval for $q$ such that when $q \in (1 - \epsilon, 1)$, $\frac{d}{dq} \left( \theta^U(P_e) - \theta_c \right) < 0$ if and only if $P_e < D_2$. Combining with $\theta_c = \theta^U(P_e)$ when $q = 1$, we obtain $\theta^U(P_e) > \theta_c$ when $q \in (1 - \epsilon, 1)$. That is $\theta_c \in [D_2, \theta^U(P_e)] \subset [D_2, \theta]$. To conclude, $\theta_c$ and $P_e$ derived above is the unique equilibrium critical cash flow and asset price in the baseline model. Thus, the creditors’ beliefs and asset buyers’ beliefs are consistent.

\[\square\]
With \( \theta = 0 \), it can be further verified that \( P_e = \frac{D_1 + D_2}{2} \) and \( D_1(D_2 - P_e) + (\theta - D_1)(P_e - D_1) = 0 \). Thus, we obtain \( \text{sgn} \left( \frac{\partial IL}{\partial \Delta} \right) = 0 \). In this case, increasing capital (increase \( \Delta \)) has no effect on a bank’s illiquidity risk.

When \( \theta > 0 \), we take the derivative \( \frac{\partial}{\partial \theta} [D_1(D_2 - P_e) + (\theta - D_1)(P_e - D_1)] = (\theta - 2D_1) \frac{\partial P_e}{\partial \theta} + (P_e - D_1) \). Recall again \( P_e = \frac{(D_1 + D_2 + \theta) + \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1 \theta}}{4} \), we can calculate \( \frac{\partial P_e}{\partial \theta} = \frac{P_e - D_1}{\sqrt{(D_1 + D_2 + \theta)^2 - 8D_1 \theta}} \).

Thus, we have

\[
(\theta - 2D_1) \frac{\partial P_e}{\partial \theta} + (P_e - D_1) = (P_e - D_1) \frac{\theta - 2D_1 + \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1 \theta}}{\sqrt{(D_1 + D_2 + \theta)^2 - 8D_1 \theta}}.
\]

As \( P_e > D_1 \), the sign of this term depends on \( \theta - 2D_1 + \sqrt{(D_1 + D_2 + \theta)^2 - 8D_1 \theta} \). When \( 2D_1 < \theta \), this term is of course larger than zero. When \( 2D_1 > \theta \), it can be verified that \( (D_1 + D_2 + \theta)^2 - 8D_1 \theta > (2D_1 - \theta)^2 \). Again, we have the term is larger than zero. When \( \theta > 0 \), we proved that

\[
\frac{\partial}{\partial \theta} [D_1(D_2 - P_e) + (\theta - D_1)(P_e - D_1)] > 0.
\]

Notice that \( D_1(D_2 - P_e) + (\theta - D_1)(P_e - D_1) = 0 \) when \( \theta = 0 \). As a result, when \( \theta > 0 \), this term is larger than zero. In the end, we have

\[
\text{sgn} \left( \frac{\partial IL}{\partial \Delta} \right) = -\text{sgn} (D_1(D_2 - P_e) + (\theta - D_1)(P_e - D_1)) < 0.
\]

Increasing capital reduces illiquidity risk in the cases where \( \theta > 0 \).

To summarize, when \( \theta = 0 \), increasing capital has no effect on illiquidity risk. When \( \theta > 0 \), increasing capital reduces illiquidity risk. But one thing should be emphasized is that increasing capital is less able to reduce illiquidity because of the “inferencing effect”.\( \square \)

**Appendix B.3  Lemma 2. The monotonicity of \( F_1(\theta) \)**

**Proof.** We start with \( F_1(\theta) \), the buyers’ expected payoff when they expecting one bank run.

With the ex post beliefs about state established, \( F_1(\theta) \) can be explicitly expressed as:

\[
F_1(\theta) = \frac{\theta - \theta_B}{(\theta - \theta_B) + (\theta - \theta_G)} \frac{\theta + \theta_B}{2} + \frac{\theta - \theta_G}{(\theta - \theta_B) + (\theta - \theta_G)} \frac{\theta + \theta_G}{2} - \frac{qD_1}{1 - \frac{D_2 - D_1}{\theta}} - \frac{qD_1 \theta}{\theta - (D_2 - D_1)}
\]

\[
= \frac{1}{2} \frac{2 \theta^2 - (\theta_B^2 + \theta_G^2)}{2 \theta - (\theta_B + \theta_G)} - \frac{qD_1 \theta}{\theta - (D_2 - D_1)}
\]

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To check the monotonicity of $F_1(\theta)$, we take the derivative:

$$
\frac{dF_1(\theta)}{d\theta} = \frac{1}{2} \left[ \frac{2\theta - (\theta_B + \theta_G)^2}{[2\theta - (\theta_B + \theta_G)]^2} + \frac{qD_1(D_2 - D_1)}{[\theta - (D_2 - D_1)]^2} \right] > 0
$$

\[ \square \]

**Appendix B.4 Proposition 4. The existence and uniqueness of $\theta^*_1$**

**Proof.** It takes two steps to prove Proposition 4. First, we prove the existence and the uniqueness of $\theta^*_1$ in the interval $[D_2, \overline{\theta}]$. Second, we prove the equilibrium cash flow $\theta^*_1 \in [\theta^L, \theta^U(P^*_1)]$ and the equilibrium price $P^*_1 \in (D_1, D_2)$. Note that $F_1(\theta)$ can be rewritten as

$$
F_1(\theta) = \omega^B_1(\theta, 1)\pi^B(\theta) + \omega^G_1(\theta, 1)\pi^G(\theta),
$$

where $\pi^s(\theta) = \frac{\theta + \theta}{2} - \frac{qD_1}{\theta - (D_2 - D_1)}$. Thus, the equilibrium condition can be also rewritten as

$$
\omega^B_1(\theta^*_1)\pi^B(\theta^*_1) + \omega^G_1(\theta^*_1)\pi^G(\theta^*_1) = 0 \tag{B.27}
$$

Step 1: We prove by continuity that there exists $\theta^*_1 \in [D_2, \overline{\theta}]$ such that $F_1(\theta^*_1) = 0$.

We value the function $F_1(\theta)$ at $\theta = D_2$. Notice that

$$
\omega^B_1(D_2) = \frac{D_2 - \theta_B}{(D_2 - \theta_B) + (D_2 - \theta_G)} > 0 \quad \text{and} \quad \omega^G_1(D_2) = \frac{D_2 - \theta_G}{(D_2 - \theta_B) + (D_2 - \theta_G)} > 0.
$$

Moreover, as $q$ sufficiently close to 1, it holds that

$$
\pi^B(D_2) = \frac{D_2 + \theta_B}{2} - qD_2 < 0 \quad \text{and} \quad \pi^G(D_2) = \frac{D_2 + \theta_G}{2} - qD_2 < 0,
$$

by the parameter assumption 1. Therefore, we have $F_1(D_2) < 0$.

Now we examine $F_1(\theta)$ at $\theta = \overline{\theta}$. Similarly, we have

$$
\omega^B_1(\overline{\theta}) = \frac{\overline{\theta} - \theta_B}{(\overline{\theta} - \theta_B) + (\overline{\theta} - \theta_G)} > 0, \quad \text{and} \quad \omega^G_1(\overline{\theta}) = \frac{\overline{\theta} - \theta_G}{(\overline{\theta} - \theta_B) + (\overline{\theta} - \theta_G)} > 0.
$$

And under our assumption 2, it holds that

$$
\pi^B(\overline{\theta}) = \frac{\overline{\theta} + \theta_B}{2} - \frac{qD_1\overline{\theta}}{\overline{\theta} - (D_2 - D_1)} > 0 \quad \text{and} \quad \pi^G(\overline{\theta}) = \frac{\overline{\theta} + \theta_G}{2} - \frac{qD_1\overline{\theta}}{\overline{\theta} - (D_2 - D_1)} > 0.
$$

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These inequalities hold when \( q \) is sufficiently close to 1 as

\[
\lim_{q \to 1^-} \pi^B(\theta) = \frac{(\theta + \theta_B)(\theta - F) - 2D_1\theta}{\theta - F} > \frac{2D_2(\theta - F) - 2D_1\theta}{\theta - F} > \frac{2F(\theta - D_2)}{\theta - D_2}
\]

Notice that \( D_2 - D_1 \) tends to \( F \) when \( q \to 1^+ \). The first inequality is by the efficiency assumption 2, \( \frac{\theta + \theta_B}{\theta - F} > D_2 \). And \( \theta > D_2 \) follows the efficiency assumption 2 as well. The proof \( \pi^G(\theta) > 0 \) of course holds. Therefore, we have: \( F_1(\theta) > 0 \). By the continuity of function \( F_1(\theta) \), there exists \( \theta_1^* \in (D_2, \theta) \) such that \( F_1(\theta_1^*) = 0 \).

Step 2: We prove \( P_1^* \in (D_1, D_2) \) and \( \theta_1^* < \theta^U(P_1^*) \).

Note that \( \theta_1^* = \frac{D_2 - D_1}{1 - qD_1/P_1^*} \) holds in equilibrium. This \( P_1^* \) is unique for \( \theta_1^* \in [D_2, \theta] \), as such \( \theta_1^* \) is unique. Moreover, \( P_1^* \) can not be equal to \( D_1 \). Otherwise, \( \theta_1^* \) can never belong to a finite region \([D_2, \theta]\).

We then consider the case when \( q \to 1^+ \): \( \lim_{q \to 1^-} \frac{D_2 - D_1}{1 - qD_1/P_1^*} = \frac{D_2 - D_1}{1 - \tilde{D}_1} \). It can be seen that \( \frac{D_2 - D_1}{1 - \tilde{D}_1} > 0 \) only if \( P_1^* > D_1 \) and \( \frac{D_2 - D_1}{1 - \tilde{D}_1} > \tilde{\theta} = D_2 \) only if \( P_1^* < D_2 \). Thus, we prove \( P_1^* \) belongs to \((D_1, D_2)\). With \( \theta_1^* = \frac{D_2 - D_1}{1 - qD_1/P_1^*} \), \( F_1(\theta_1^*) = 0 \) can be also rewritten as \( \omega^B(P_1^*) \cdot \pi^B(P_1^*) + \omega^G(P_1^*) \cdot \pi^G(P_1^*) = 0 \). Hence, such \( P_1^* \) indeed makes the asset buyers earn zero profit when one bank run is observed.

Similar as in Appendix B.1, the derivative \( \lim_{q \to 1^-} \frac{\partial}{\partial q} \left[ \theta^U(P_1^*) - \tilde{\theta}(P_1^*) \right] = \frac{P_1^* - D_2}{P_1^*} \). Having proved \( P_1^* < D_2 \), \( \theta^U(P_1^*) > \tilde{\theta}(P_1^*) \) when \( q \in (1 - \epsilon, 1) \). That is \( \theta_1^* \in [D_2, \theta^U(P_1^*)] \subset [D_2, \theta] \). Recall Appendix A, such \( \theta_1^* = \frac{D_2 - D_1}{1 - qD_1/P_1^*} \) is indeed a threshold equilibrium given asset price \( P_1^* \).

To summarize, the unique price \( P_1^* \in (D_1, D_2) \) indeed makes the asset buyers make zero profit, and no incentive to deviate. And the unique \( \theta_1^* \in [D_2, \theta] \) is indeed an equilibrium critical cash flow. Combine those two, the equilibrium \( \{\theta_1^*, P_1^*\} \) exists and is unique when one bank run is observed.

\[ \square \]

**Appendix B.5 Lemma 3. The monotonicity of \( F_2(\theta) \)**

**Proof.** We show the monotonicity of \( F_2(\theta) \), the buyers’ expected payoff when they expecting two bank runs. We write explicitly function \( F_2(\theta) \) as:

\[
F_2(\theta) = \frac{1}{2} \left[ \frac{(\theta - \theta_B)^2(\theta + \theta_B) + (\theta - \theta_G)^2(\theta + \theta_G)}{(\theta - \theta_B)^2 + (\theta - \theta_G)^2} \right] - \frac{qD_1\theta}{\theta - (D_2 - D_1)}
\]
Appendix B.6 Proposition 5. The existence and uniqueness of $\theta_2^*$

Proof. We follow the same argument as the proof in Appendix B.4. Similarly, the equilibrium condition can be expressed as

$$F_2(\theta_2^*) = \omega_2^B(\theta_2^*)\pi^B(\theta_2^*) + \omega_2^G(\theta_2^*)\pi^G(\theta_2^*) = 0 \quad (B.28)$$

To check the step 1. Notice that:

$$\omega_2^B(D_2) = \frac{(D_2 - \theta_1)^2}{(D_2 - \theta_b)^2 + (D_2 - \theta_G)^2} > 0, \quad \omega_2^G(D_2) = \frac{(D_2 - \theta_G)^2}{(D_2 - \theta_b)^2 + (D_2 - \theta_G)^2} > 0.$$  

Moreover,

$$\omega_2^B(\bar{\theta}) = \frac{(\bar{\theta} - \theta_b)^2}{(\bar{\theta} - \theta_b)^2 + (\bar{\theta} - \theta_G)^2} > 0, \quad \omega_2^G(\bar{\theta}) = \frac{(\bar{\theta} - \theta_G)^2}{(\bar{\theta} - \theta_b)^2 + (\bar{\theta} - \theta_G)^2} > 0.$$
The sign of function $F_2(\theta)$ depends on $\pi^B(\theta)$ and $\pi^G(\theta)$, which have the same definitions as in Appendix B.4. We have already showed that: $\pi^B(D_2) < 0, \pi^B(\bar{\theta}) > 0$ and $\pi^G(D_2) < 0, \pi^G(\bar{\theta}) > 0$. Thus we can again claim:

$$F_2(D_2) < 0 \quad \text{and} \quad F_2(\bar{\theta}) > 0.$$ 

By the continuity of $F_2(\theta)$, there exists a $\theta^*_2 \in (D_2, \bar{\theta})$ satisfying $F_2(\theta^*_2) = 0$. Then by Lemma 1, $\theta^*_2$ necessarily belongs to $(D_2, \theta^U(P_2^*))$ with $P_2^* = \frac{q_{D_2} \theta^*_2}{\theta(D_2 - D_1)}$.

Since $F_2$ is monotonically increasing in $\theta$, the uniqueness of this $\theta^*_2$ is again guaranteed.

The equilibrium $\{\theta^*_2, P_2^*\}$ exists and is unique.

Then step 2 follows exactly the procedure as in Appendix B.4, we thus omit it. □

**Appendix B.7 Proposition 6. Financial contagion**

**Proof.** The proof hinges on the monotonicity of two ratios

$$\frac{\omega^B(\theta)}{\omega^G(\theta)} = \frac{(\theta - \theta_B)^2}{(\theta - \theta_G)^2} \quad \text{and} \quad \frac{\pi^G(\theta)}{\pi^B(\theta)} = \frac{\theta + \theta_G}{2} - P(\theta).$$

The first is a conditional likelihood ratio and the second is a payoff ratio. It can be shown both ratios are strictly monotonically decreasing in $\theta$, that is

$$\frac{d}{d\theta} \left( \frac{\omega^B(\theta)}{\omega^G(\theta)} \right) = \frac{-2(\theta - \theta_B)(\theta_G - \theta_B)}{(\theta - \theta_G)^2} < 0$$

$$\frac{d}{d\theta} \left( \frac{\pi^G(\theta)}{\pi^B(\theta)} \right) = \frac{[\frac{1}{2} - P'(\theta)](\theta_B - \theta_G)}{[\frac{\theta + \theta_G}{2} - P(\theta)]} < 0$$

We focus on the interior realization of cash flow, then $\theta > \theta_G$. And remember $P'(\theta) < 0$ from the Appendix B.4.

Furthermore, notice that for $\omega^B(\theta)/\omega^G(\theta) > 1$, we have

$$\frac{\omega^B(\theta)}{\omega^G(\theta)} < \left[ \frac{\omega^B(\theta)}{\omega^G(\theta)} \right]^2 = \frac{\omega^B(\theta)}{\omega^G(\theta)}$$

(B.29)

Now we prove by contradiction. Suppose $\theta^*_1 > \theta^*_2$. By the monotonicity of $\pi^G(\theta)/\pi^B(\theta)$, we will have

$$\frac{\pi^G(\theta^*_1)}{\pi^B(\theta^*_1)} < \frac{\pi^G(\theta^*_2)}{\pi^B(\theta^*_2)}.$$
By the equilibrium conditions (B.27) and B.28, we have
\[
\frac{\pi^G(\theta^*_1)}{\pi^B(\theta^*_1)} = -\frac{\omega^B(\theta^*_1)}{\omega^G(\theta^*_1)} \quad \text{and} \quad \frac{\pi^G(\theta^*_2)}{\pi^B(\theta^*_2)} = -\frac{\omega^B(\theta^*_2)}{\omega^G(\theta^*_2)},
\]
which implies
\[
\frac{\omega^B(\theta^*_2)}{\omega^G(\theta^*_2)} < \frac{\omega^B(\theta^*_1)}{\omega^G(\theta^*_1)}.
\]
By condition (B.29), we know
\[
\frac{\omega^B(\theta^*_2)}{\omega^G(\theta^*_2)} < \frac{\omega^B(\theta^*_1)}{\omega^G(\theta^*_1)} < \frac{\omega^B(\theta^*_2)}{\omega^G(\theta^*_2)}.
\]
But this contradicts the monotonicity of \(\omega^B(\theta)/\omega^G(\theta)\). Therefore, we prove \(\theta^*_2 > \theta^*_1\). \(\square\)

Appendix B.8 Lemma 4. Regulator’s break even price \(P^*_A\)

Proof. By inserting (21) into (22), one can obtain the following equation.
\[
4(P_A)^2 - [2(D_2 - D_1) + 4qD_1 + (\theta_B + \theta_G)]P_A + qD_1(\theta_B + \theta_G) = 0
\]
The positive solution of this quadratic function is
\[
P^*_A = \frac{[2(D_2 - D_1) + 4qD_1 + (\theta_B + \theta_G)] + \sqrt{[2(D_2 - D_1) + 4qD_1 + (\theta_B + \theta_G)]^2 - 16qD_1(\theta_B + \theta_G)}}{8}
\]
Following the proof in Appendix B.1, we can check that \(P^*_A \in (D_1, D_2)\). Moreover, we can also check that the regulator does not have profitable deviation by unilaterally bid higher price than \(P^*_A\). \(\square\)

Appendix B.9 Proposition 7. Asset purchase

Proof. Recall that \(\theta^*_1\) solves \(F_1(\theta^*_1) = 0\). \(F_1(\theta)\) can be rewritten as
\[
F_1(\theta) = \frac{1}{2} \frac{\theta_B + \theta}{2} + \frac{1}{2} \frac{\theta_G + \theta}{2} - \frac{qD_1 \theta}{\theta - (D_2 - D_1)} - \frac{(\theta_G - \theta_B)^2}{4[(\theta - \theta_G) + (\theta - \theta_B)]}
\]
While, we can define
\[
F_A(\theta) = \frac{1}{2} \frac{\theta_B + \theta}{2} + \frac{1}{2} \frac{\theta_G + \theta}{2} - \frac{qD_1 \theta}{\theta - (D_2 - D_1)},
\]

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where $F_A(\theta_A^*) = 0$. Insert $\theta^*$ into it, we have

$$F_1(\theta_A^*) = F_A(\theta_A^*) - \frac{(\theta_G - \theta_B)^2}{4[(\theta - \theta_G) + (\theta - \theta_B)]} = -\frac{(\theta_G - \theta_B)^2}{4[(\theta - \theta_G) + (\theta - \theta_B)]} < 0$$

Recall again $F_1(\theta)$ is increasing in $\theta$. We have $\theta_A^* < \theta^*$. Then $P_A^* > P_1^*$ immediately follows. □

Appendix B.10 Lemma 5. Regulatory transparency and bank runs

Proof. We solve here only for the equilibrium in state $s = G$. The equilibrium under $s = 1$ can be solved with the same procedure. The equilibrium is determined by a system of two equations:

$$\begin{cases}
\theta^G_c = \frac{D_2 - D_1}{p^G} \\
p^G_e = \frac{\theta^G_c + \theta_G}{2}
\end{cases}$$

Solving the system of equations as in the Appendix B, we have the equilibrium critical cash flow and the endogenous fire-sale price:

$$\theta^G_c = \frac{(D_2 - D_1) + 2qD_1 - \theta_G + \sqrt{[(D_2 - D_1) + 2qD_1 - \theta_G]^2 + 4(D_2 - D_1)\theta_G}}{2}$$

$$p^G_e = \frac{(D_2 - D_1) + 2qD_1 + \theta_G \pm \sqrt{[(D_2 - D_1) + 2qD_1 + \theta_G]^2 - 8qD_1\theta_G}}{4}$$

When $q$ is sufficiently close to 1, we have

$$\theta^G_c = \frac{(D_1 + D_2 - \theta_G) + \sqrt{[D_1 + D_2 - \theta_G]^2 + 4F\theta_G}}{2}$$

$$p^G_e = \frac{(D_1 + D_2 + \theta_G) + \sqrt{[D_1 + D_2 + \theta_G]^2 - 8D_1\theta_G}}{4}$$

It is straightforward to check that $\theta^G_c \in (D_2, \theta]$ as in Appendix B.2. □

Appendix B.11 Proposition 8. Regulatory transparency and illiquidity

Proof. We start by proving $\theta^G_c < \theta^*_1$. Recall that $F_1(\theta^*_1) = 0$ and $F_1(\theta)$ is monotonically increasing. So $\theta^G_c < \theta^*_1$ will hold if and only if $F_1(\theta^G_c) < 0$. To proceed, we write $F_1(\theta)$ explicitly

$$F_1(\theta) = \frac{\theta - \theta_B}{(\theta - \theta_B) + (\theta - \theta_G)} \frac{\theta + \theta_B}{2} + \frac{\theta - \theta_G}{(\theta - \theta_B) + (\theta - \theta_G)} \frac{\theta + \theta_G}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)}.$$
We can rewrite $F_1(\theta)$ as follows:

$$F_1(\theta) = \frac{\theta - \theta_B}{(\theta - \theta_B) + (\theta - \theta_G)} \left[ \frac{\theta + \theta_B}{2} - \frac{\theta + \theta_G}{2} \right] + \frac{\theta + \theta_G}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)}.$$ 

We then evaluate $F_1(\theta)$ at $\theta_{e^*}^G$, that is

$$F_1(\theta_{e^*}^G) = -\frac{\theta_{e^*}^G - \theta_B}{(\theta_{e^*}^G - \theta_B) + (\theta_{e^*}^G - \theta_G)} \frac{\theta_G - \theta_B}{2} < 0.$$ 

We then evaluate $F_1(\theta)$ at $\theta_{e^*}^B$, that is

$$F_1(\theta_{e^*}^B) = -\frac{\theta_{e^*}^B - \theta_B}{(\theta_{e^*}^B - \theta_B) + (\theta_{e^*}^B - \theta_G)} \frac{\theta_G - \theta_B}{2} < 0.$$ 

Recall that the term $\frac{\theta_{e^*}^G + \theta_{e^*}^B}{2} - \frac{qD_1\theta_{e^*}^G}{\theta - (D_2 - D_1)} = \frac{\theta_{e^*}^G + \theta_{e^*}^B}{2} - P_G = 0$. Then we have $\theta_{e^*}^G < \theta_{e^*}^B$.

We then prove $\theta_{e^*}^G > \theta_{e^*}^B$. Recall that $F_2(\theta_{e^*}^B) = 0$ and $F_2(\theta)$ is monotonically increasing. So $\theta_{e^*}^G > \theta_{e^*}^B$ will hold if and only if $F_2(\theta_{e^*}^B) > 0$. Similarly, we can write $F_2(\theta)$ as

$$F_2(\theta) = \frac{(\theta - \theta_G)^2}{(\theta - \theta_B)^2 + (\theta - \theta_G)^2} \frac{\theta_G - \theta_B}{2} + \frac{\theta + \theta_B}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)}.$$ 

We evaluation $F_2(\theta)$ at $\theta_{e^*}^B$, and for the similar argument

$$F_2(\theta_{e^*}^B) = \frac{(\theta_{e^*}^B - \theta_G)^2}{(\theta_{e^*}^B - \theta_B)^2 + (\theta_{e^*}^B - \theta_G)^2} \frac{\theta_G - \theta_B}{2} > 0.$$ 

Then we have $\theta_{e^*}^B > \theta_{e^*}^B$.

\[\square\]

**Appendix B.12 Proposition 9. Socially undesirable disclosure**

**Proof.** It can be seen easily $SC^{AP} < SC^{RT}$ if and only if $\theta^* < \frac{\theta_{e^*}^G + \theta_{e^*}^B}{2}$. Consider the auxiliary function

$$G(\theta) = \frac{2qD_1\theta}{\theta - (D_2 - D_1)} - \theta.$$ 

Then $\theta^*$ satisfies $G(\theta^*) = \frac{\theta_{e^*}^G + \theta_{e^*}^B}{2}$. $\theta_{e^*}^G$ and $\theta_{e^*}^B$ combined satisfy $\frac{1}{2}G(\theta_{e^*}^G) + \frac{1}{2}G(\theta_{e^*}^B) = \frac{\theta_{e^*}^G + \theta_{e^*}^B}{2}$. Together we obtain

$$G(\theta^*) = \frac{1}{2}G(\theta_{e^*}^G) + \frac{1}{2}G(\theta_{e^*}^B).$$
It is fairly easy to check that $G' = \frac{2qD_1(D_2-D_1)}{[\theta-(D_2-D_1)]^2} < 0$ and $G''(\theta) = \frac{4qD_1(D_2-D_1)(\theta-(D_2-D_1))}{[\theta-(D_2-D_1)]^4} > 0$, thus $G$ is a decreasing convex function. We further have

$$G(\theta^a) = \frac{1}{2}G(\theta^G_e) + \frac{1}{2}G(\theta^B_e) > G(\frac{\theta^G_e + \theta^B_e}{2})$$

Lastly, because the function $G$ is decreasing, we obtain $\theta^*_A < \frac{\theta^G_e + \theta^B_e}{2}$. The social cost due to illiquidity is lower than the regulator chooses to implement the asset purchase program.