Modeling Credit Contagion via the Updating of Fragile Beliefs

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Abstract

We propose a general equilibrium framework for pricing defaultable bonds that are subject to contagion-risk when the representative agent has fragile beliefs (Hansen and Sargent (2009)). We identify conditions for which the marginal utility of the agent with fragile beliefs generate time-consistent state prices. Even though capturing contagion implies that our intensity-based model falls outside of the “doubly stochastic” framework, bond prices remain tractable, in turn facilitating empirical investigation. We apply this model to sovereign countries in the European Union (EU). The model can justify large and highly correlated credit spreads even when default probabilities and correlations in macroeconomic fundamentals are low.
1 Introduction

During the recent (and ongoing) Euro-crisis, the risk of contagion has often been cited as one of the major drivers of sovereign credit spreads. Indeed, typing the words “contagion and Euro” into a Google search returns over 750,000 results, many of which refer to articles from the financial press that relate changes in sovereign spreads of European nations to the risk or ‘fear’ of contagion.¹ This dialogue raises many important questions, including:²

- What is contagion risk, and what are its economic sources?
- Is there a risk-premium associated with contagion risk, and if so, what is its impact on sovereign spreads?
- To what extent is the co-movement in sovereign spreads driven by contagion risk and its risk-premium?

In this paper we propose a general equilibrium framework that offers an economic justification for why contagion may be an important driver of both the level and the correlation of sovereign spreads. The model contains two important ingredients. First, we assume there is a hidden state of nature whose actual value, if known, would impact the expected consumption growth and probability of default across all countries. Second, we assume that the preferences of the representative agent in our economy are described by a “fragile beliefs” utility function, as specified in Hansen and Sargent (2010). Together, these two ingredients can capture large sovereign spreads even if expected losses due to default are relatively small. Furthermore, the model can justify significant correlation in spreads even if common movements in macroeconomic fundamentals are relatively modest.

In addition to capturing these empirical observations, our paper makes two additional theoretical contributions. First, even though capturing contagion implies that our intensity-based model falls outside of the so called “doubly stochastic” framework, bond prices remain tractable, in turn facilitating empirical investigation. In contrast, most of the previous literature specifies a “doubly-stochastic” (or Cox process) framework, where the default event is


²Of course, this is not uncontroversial. The Google search also returns the following title, by the eminent economist, John Cochrane: ‘Contagion’ and other Euro myths’ WSJ, Dec 2, 2010. He writes: “The bailout is being justified on grounds of containing ”contagion.” This is nonsense. The notion is that news of an Irish restructuring would scare investors in Spanish bonds, who would start looking at Spain’s ability to repay its debts and then demand higher interest rates. But haven’t investors in Spanish bonds already noticed that there’s a bit of a problem? And wouldn’t news of a giant bailout make these investors question Spanish finances as much as would news of debt restructuring?”

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conditionally independent of the state variables driving the intensity processes. Such doubly stochastic processes inherently preclude any type of contagion due to defaults. To understand this intuitively, note that in our framework, when one sovereign defaults, agents will Bayesian update their beliefs, placing a higher probability that the economy is in the ‘bad’ hidden state. This updating will cause agents to increase the default intensities for all other countries, creating a ‘contagious’ market-wide increase in credit spreads. This type of contagious mechanism cannot be captured in a doubly stochastic framework.

Our second theoretical contribution is to identify conditions for which the fragile beliefs preferences provide a time-consistent framework. That is, intuitively, we identify conditions for which fragile beliefs equate the present value of future dividends of a given long-lived asset to its present value of next period’s sum of (price + dividend). We find we need two types of restrictions. First, we show that if the agent has preference for both robustness and fragility, the general equilibrium price processes will not be time-consistent. Instead, one either needs to turn off preference for robustness by combining fragile beliefs with conditionally time-separable preferences, or turn off fragility by assuming no hidden state. Second, even when fragile beliefs are combined with conditionally time-separable preferences, we find that certain ‘entropy-adjusted weights’ need to be state-independent for the framework to produce time-consistent prices.

In our model, the ability of a country to repay its debt (which is captured through a hazard rate/intensity process) is a function of the probabilities associated with the hidden state. Hence, in equilibrium, credit spreads will depend on the posterior probability of the hidden state assessed by market participants. This updating of beliefs will generate correlations in credit spreads that are significantly higher than if these spreads were functions only of the macroeconomic conditions. Furthermore, in contrast to the standard framework with time-separable preferences, an agent with fragile beliefs will demand compensation for exposure to uncertainty about the true value of the hidden state. This will increase spreads and expected returns on sovereign debt.

We estimate our model using data on sovereign CDS from February 2004 to June 2010 for 11 Euro-zone countries. We use a two stage approach. First, we follow the existing literature on sovereign risk to identify a list of variables that have been shown to predict a country’s ability or propensity to repay its debt. We summarize the information in this large list of variables using a dynamic principal component approach of Stock and Watson (1989, 1991).

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Since the data is observed at mixed frequencies, we follow the filtering approach of Aruoba, Diebold and Scotti (2008). Assuming a multivariate AR(1) process, this approach provides us with both an estimate of the time series of these underlying variables and a parameter vector that captures their dynamics. Conditional default intensities are specified to be linear functions of these state variables.

We then use panel data (time-series and cross-section) of sovereign CDS spreads and time series of consumption, default events and other ‘signals’ to identify model parameters and the time series of the filtered posterior probability of the hidden state. This calibration provides us with a characterization of the sensitivity of the various European countries to the posterior probability state variable, which captures the contagion risk in our model. Our model delivers an estimate of the conditional contagion risk-premium for each member country, which we plot in time series. We thus obtain a CDS-implied index of the level of contagion risk-premium assessed by the market participants.

Our paper builds on and combines two important strands of literature: event risk and Bayesian updating of beliefs. Conditions for which jump-to-default is not priced have been investigated by Jarrow, Lando and Yu (2005). They demonstrate that under some standard APT-like assumptions\(^4\) that jump-to-default risk will not be priced if the default process is assumed to follow a so-called “doubly stochastic” (or Cox) process. However, recent empirical findings question this doubly-stochastic assumption. For example, Das et al. (2006, 2007) report that the observed clustering of defaults in actual data are inconsistent with this assumption. Duffie et al (2009) use a fragility-based model similar to ours to identify a hidden state variable consistent with a contagion-like response. Note that the focus of these papers is on estimating the empirical default probability, whereas our focus is on pricing. Jorion and Zhang (2007) find contagious effects at the industry level.\(^5\)

Other papers investigating event risk include Jarrow and Yu (2001), who also provide a model where the default of one firm affects the intensity of another. However, their model remains tractable only for a “small” number \(N\) of firms exposed to contagion-risk (e.g., JY investigate only \(N = 2\)). In contrast, our model remains tractable regardless of the number of firms that share in the contagious response. Such a framework is necessary for our purposes since we want to investigate how large the jump-to-default risk premium can be when the number of sovereign entities is large.\(^6\) Models of credit risk embedded within a macroeconomic

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\(^4\)These assumptions include: i) each firm constitutes a small fraction of the economy, ii) a finite fraction of firms do not default simultaneously, and iii) marginal investors can diversify their bond portfolio holdings.

\(^5\)Related, also see Jorion and Zhang (2009), Lando and Nielsen (2009).

\(^6\)Collin-Dufresne, Goldstein and Hugonnier (2004) simplify the bond pricing formula of Duffie, Schroeder and Skiadis (1996). Note, however, that the formula itself does not identify a tractable framework for pricing.
setting include David (2008), Chen, Collin Dufresne and Goldstein (2009), Chen (2010), and Bhamra, Kuehn and Strebulaev (2011).

Our approach shares many common features with those in the learning and contagion literature (Detemple (1986), Feldman (1989), David (1997), Veronesi (1999, 2000))\(^7\). As in these papers, the representative agent in our economy learns about a hidden state from observing aggregate consumption. However, in addition in our model the agent learns from individual firms’ defaults history.\(^8\) Further, we identify a time-consistent model of a representative agent that has fragile beliefs (Hansen and Sargent (2009), Hansen (2007)). Time consistency allows us to price securities with long-dated cash flows in a tractable manner. This framework naturally generates a flight-to-quality (i.e., a drop in risk free rates) caused by an unexpected default, consistent with observation.

Our information-based mechanism for contagion is similar to that proposed by King and Wadhwani (1990) and Kodres and Pritsker (2002), who investigate contagion across international financial markets. There is also a large empirical literature that studies contagion in equity markets (e.g., Lang and Stulz (1992)) and in international finance (e.g., Bae, Karolyi and Stulz (2003)). Theocharides (2007) investigates contagion in the corporate bond market and finds empirical support for information-based transmission of crises.

The rest of the paper is as follows. In Section 2, we propose an intensity-based model of sovereign risk with a hidden state and show how beliefs are updated from observing default events and other signals. In Section 3 we investigate the pricing implications of this model by incorporating it in a general equilibrium framework where the representative agent has fragile beliefs. We then calibrate the model in Section 4 using six years of sovereign CDS prices. We conclude in Section 5. In the Appendix, we identify necessary conditions for fragile beliefs preferences to generate time consistent price processes.

### 2 Updating Beliefs by Observing Default Processes

Consider an economy where the true state of nature \(\tilde{S}\) is unknown and can be in any one of \(s \in (1, M)\) states. At date-\(t\), investors do not know what state the economy is in, but form a prior \(\pi_s(t) \equiv \text{Prob}(\tilde{S} = s|\mathcal{F}_t)\), where \(\mathcal{F}_t\) is the investors’ information set at date-\(t\). In this economy there are \(N\) defaultable entities (firms, countries) indexed by \(i \in (1, N)\) with contagion risk. Other models of contagion include Davis and Lo (2001), Schönbucher and Schubert (2001), and Giesecke (2004).

\(^7\)Also see Ait-Sahalia, Cacho-Diaz and Laeven (2010), Benzoni, Collin-Dufresne and Goldstein (2011)

\(^8\)I addition, a minor technical difference between these papers and ours is that they use results on filtering theory for diffusions, whereas in our case information is revealed through jump-diffusion processes.
random default times $\tau_i$ driven by point processes characterized by default intensities. In particular, conditional upon being in state-$s$, the probability of default over the next interval $dt$ is expressed via

$$\Pr \left[ d1_{\{\tau_i < t\}} = 1 \mid S = s, F_i \right] = E \left[ d1_{\{\tau_i < t\}} \mid S = s, F_i \right] = \lambda_{is}(t^-) 1_{\{\tau_i > t\}} dt. \quad (1)$$

That is, we can interpret $\lambda_{is}(t^-)$ as the date-$t$ default intensity for firm-$i$ conditional upon being in state-$s$. Below, we will assume that, conditioning both on the state-$s$ and the paths $\lambda_{is}(t^-)|_{t=0}^{T}$ for some distant future date-$T$, the default events across firms are independent. In technical terms, we are assuming a doubly-stochastic, or Cox-process conditional upon being in a particular state-$s$.\(^9\) We emphasize, however, because agents do not know the correct state-$s$, our model falls outside of the Cox-process framework, as will be made clear below in, for example, equation (2).

Since investors do not know the actual state of nature, their estimate of the actual default intensity $\lambda_i^P(t^-)$ is defined implicitly through

$$\lambda_i^P(t^-) 1_{\{\tau_i > t\}} dt = E \left[ d1_{\{\tau_i < t\}} \mid F_i \right] = \sum_{s=1}^{M} \pi_s(t) E \left[ d1_{\{\tau_i < t\}} \mid S = s, F_i \right] = \sum_{s=1}^{M} \pi_s(t) \lambda_{is}(t^-) 1_{\{\tau_i > t\}} dt. \quad (2)$$

Thus, conditional on investors’ information, the default intensity of firm $i$ is equal to a weighted average of the conditional default intensities:

$$\lambda_i^P(t^-) = \sum_{s=1}^{M} \pi_s(t) \lambda_{is}(t^-). \quad (3)$$

We assume that investors continuously update their estimates of the $\{\pi_s(t)\}$ conditional upon whether or not they observe a default event during the interval $dt$. A direct application of Theorem 19.6 page 332 in Lipster and Shiryaev (see also their example 1 p. 333) gives the updating equation for $\pi_s(t)$:\(^{10}\)

$$\frac{d\pi_s(t)}{\pi_s(t^-)} = \sum_{i=1}^{N} \left( \frac{\lambda_i^P(t^-)}{\lambda_i^P(t^-)} - 1 \right) dM_i(t), \quad (4)$$

\(^9\)See, for example, Lando (1998)

\(^{10}\)We also provide a heuristic derivation of this result based solely on Bayes’ rule in Appendix ??.

We note that it would be straightforward to extend the model to allow for unobserved random transitions between states.
where
\[
\frac{dM_i(t)}{dt} \equiv \left( \mathbf{1}_{\{\tau_i \leq t\}} - \bar{\lambda}^P_i(t) \mathbf{1}_{\{\tau_i > t\}} \right) dt.
\] (5)

We note that this process has many intuitive properties. First, if the prior \( \pi_s(t) = 1 \) for some state-s (and thus \( \pi_{s'}(t) = 0 \) for all other \( s' \)), then there is no updating. That is, in an economy where the agents know for sure the intensity of the firms, then there is no learning to be done. Second, when no default is observed over an interval \( dt \), then investors revise downward the ‘high-default’ states of nature (i.e., those \( s \) with \( \lambda_s(t) > \bar{\lambda}^P_i(t) \)), and in turn revise upward the ‘low-default’ states of nature (i.e., those \( s \) with \( \lambda_s(t) < \bar{\lambda}^P_i(t) \)). Conversely, when a default is observed over an interval \( dt \), investors revise upward those high-default states of nature, and in turn revise downward those low-default states of nature. Third, note that \( \pi_s(t) \equiv \mathbb{E} \left[ \tilde{S} = s \middle| \mathcal{F}_t \right] \) is a \( P \)-martingale in that \( \mathbb{E}_t[d\pi_s(t)] = 0 \), as can be seen from equations (2), (4) and (5).

2.1 Updating also from continuous information

In addition to observing country default processes, investors also observe continuous signals that provide information about the state. Specifically, we assume that investors observe \( K \) signals with dynamics:
\[
\frac{d\Omega_k(t)}{dt} = \mu_{k,s} dt + \sigma_k dZ^s_k(t),
\] (6)
where the \( \{dZ^s_k(t)\} \) are independent Brownian Motions conditional on being in state-\( s \) in that \( \mathbb{E} [dZ^s_k(t) | s] = 0 \). It thus follows that the unconditional drift is
\[
\bar{\mu}_k(t) = \frac{1}{dt} \mathbb{E}_t \left[ d\Omega_k(t) \right] \\
= \sum_s \pi_s(t) \frac{1}{dt} \mathbb{E}_t \left[ d\Omega_k(t) | s \right] \\
= \sum_s \pi_s(t) \mu_{k,s}.
\] (7)

Hence, we can also express signal dynamics as
\[
\frac{d\Omega_k(t)}{dt} = \bar{\mu}_k(t) dt + \sigma_k dZ^s_k(t),
\] (8)
where the Brownian motion satisfies \( \mathbb{E} [dZ^s_k(t)] = 0 \).

It is well-known (see, for example, David (1998), Veronesi (????)) that the updating equation for the posterior probability of the state from this continuous information is given
by

\[
\frac{d\pi_s(t)}{\pi_s(t)} = \sum_{k=1}^{K} \left( \frac{\mu_{k,s} - \bar{\mu}_k(t)}{\sigma_k} \right) dZ_k(t).
\] (9)

Given that the agent observes both continuous signals \(d\Omega_k(t)\) and defaults \(d1_{r_i > t}\), it follows that the updating equation follows:

\[
\frac{d\pi_s(t)}{\pi_s(t^-)} = \sum_{i=1}^{N} \left( \frac{\lambda_{i,s}(t^-)}{\lambda_i(t^-)} - 1 \right) dM_i(t) + \sum_{k=1}^{K} \left( \frac{\mu_{k,s} - \bar{\mu}_k(t)}{\sigma_k} \right) dZ_k(t).
\] (10)

### 2.2 Model for intensity conditional on the state

We specify the default intensity of entity \(i \in (1, N)\) conditional on being in state \(s\) as:

\[
\lambda_{is}(t) = \alpha_{ts} + \beta_{is}' X(t),
\] (11)

where \(X(t)\) is a state vector which contains both country specific variables as well as a common variables. We specify \(X(t)\) to follow a multi-dimensional Gaussian affine process.

\[
dX(t) = [\psi - \kappa X(t)] dt + \Sigma dW(t),
\] (12)

where without loss of generality we specify the volatility matrix \(\Sigma\) to be lower diagonal.

Up to this point, the state variable dynamics have been specified under the historical measure. In the following section we address the issue of pricing defaultable securities in the presence of contagion risk and systematic jump risk.

### 3 General Equilibrium with Fragile Beliefs

#### 3.1 Endowment Process

For each state-\(s\), we specify the aggregate (log-) endowment process (which is owned by the representative agent) as

\[
d\log y = \mu_s dt + \sigma dZ^s_0,
\] (13)

where the Brownian motions \(dZ^s_0\) satisfy \(E[dZ^s_0|s] = 0\). Below, it will be convenient to also express the (log-) endowment process as

\[
d\log y = \pi^P(t) dt + \sigma dZ_0,
\] (14)

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11 Again, a heuristic proof is given in the appendix.
12 Note that, due to this assumption, the intensities in equation (11) can go negative. Empirically, this does not occur in our samples, so we do not think this concern significantly impacts our results.
where $\tilde{P}(t) = \sum_s \pi_s(t) \mu_s$, and where $dZ_0$ is a Brownian motion, hence $\mathbb{E}[dZ_0 | F_t] = 0$. Combining equations (13)-(14), we find for all $s$:
\[ \begin{align*}
   dZ_0 &= dZ_s^0 + \left( \frac{\mu_s - \tilde{P}(t)}{\sigma} \right) dt. 
\end{align*} \] (15)

### 3.2 Preferences

- Conditional upon being in state $s \in (1, M)$, the agent has logarithmic-preferences. That is, the agent ranks consumption lotteries in state-$s$ according to the (state contingent) index $V(\{C(\cdot)\}|F_0, s)$, which satisfies:
\[
   V(\{C(\cdot)\}|F_0, s) = \mathbb{E} \left[ \int_0^\infty \beta e^{-\beta t} \log C(t) \mid F_0, s \right] = \int_0^\infty \beta e^{-\beta t} \int dF_t \log C(F_t) \pi(F_t | F_0, s). \] (16)

- To rank consumption streams unconditionally, the agent displays fragile beliefs. That is, the agent combines the conditional utility indices using a preference for robustness parameter $\zeta$ via:
\[
   V(\{C(\cdot)\}|F_0) = \min_{\{\xi_s(0)\} > 0} \left\{ \sum_{s=1}^M \pi_s(0) \left( \xi_s(0) V(\{C(\cdot)\}|F_0, s) + \zeta \xi_s(0) \log \xi_s(0) \right) \right\}, \] (17)
subject to the constraint
\[
   1 = \sum_s \pi_s(0) \xi_s(0). \] (18)

Solving for the constrained minimization, we find
\[
   \xi_s(0) = \frac{e^{-\left(\frac{1}{\zeta}\right)V(\{C(\cdot)\}|F_0, s)}}{\sum_{s'} \pi_{s'}(0) e^{-\left(\frac{1}{\zeta}\right)V(\{C(\cdot)\}|F_0, s')}}. \] (19)

Plugging this back into equation (17), we find that preferences simplify to
\[
   V(\{C(\cdot)\}|F_0) = \left\{ -\zeta \log \left[ \sum_{s=1}^M \pi_s(0) e^{-\left(\frac{1}{\zeta}\right)V(\{C(\cdot)\}|F_0, s)} \right] \right\}. \] (20)

It is worth noting that if the agent chooses to consume the endowment stream (which will ultimately be the equilibrium in this exchange economy), we find
\[
   V(\{y(\cdot)\}|F_0, s) = \log y(0) + \frac{\mu_s}{\beta}. \] (21)
Under this scenario, the fragility parameters 

\[ \xi_s(t) = \frac{e^{-(\sum s' \pi_s(t) e^{-\frac{\beta^s}{\pi}})}}{\sum s' \pi_s(t) e^{-\frac{\beta^s}{\pi}}} \]  

(22)

are independent of the state variables \( X(t) \), and hence change over time only through their dependence on the probabilities \( \{ \pi_s(t) \} \). This feature is important for the model to be time-consistent.

Given these preferences, and assuming complete markets, the representative agent chooses her consumption stream to maximize her utility subject to the budget constraint

\[ 0 = \int_0^\infty dt \int dF_t A(F_t|F_0) [y(F_t) - C(F_t)] \]  

(23)

where the \( A(F_t|F_0) \) are the Arrow/Debreu prices (which the representative agent takes as exogenously specified). The agent’s first order conditions with respect to consumption across all states of nature \( C(F_t) \) imply:

\[ \lambda A(F_t|F_0) = \sum_s \pi_s(0) \xi_s(0) e^{-\beta t} \pi(F_t|F_0, s) \frac{1}{C(F_t)} \]  

(24)

where the Lagrange multiplier \( \lambda \) can be determined by taking the limit \( F_t \Rightarrow F_0 \).

\[ \lambda = \frac{\beta}{C(F_0)} \]  

(25)

Combining these last two equations, we find that the optimal consumption bundle satisfies

\[ A(F_t|F_0) = \sum_s \pi_s(0) \xi_s(0) e^{-\beta t} \pi(F_t|F_0, s) \frac{C(F_0)}{C(F_t)} \]  

(26)

### 3.3 Equilibrium

In this endowment economy, for markets to clear, it must be that Arrow/Debreu prices adjust until optimal consumption is equal to exogenous endowment process state-by-state. Thus, we find

\[ A(F_t|F_0) = \sum_s \pi_s(0) \xi_s(0) e^{-\beta t} \pi(F_t|F_0, s) e^{-[\log y(F_t) - \log y(F_0)]} \]  

(27)

Combining equations (13) and (27), we can express the Arrow/Debreu prices as

\[ A(F_t|F_0) = \sum_s \pi_s(0) \xi_s(0) e^{-\beta t} \pi(F_t|F_0, s) \exp \{- \left[ \mu_t + \sigma Z^s_t(F_t) \right] \} \]  

(28)
At this point it is convenient to introduce for each state-s a process that is most readily interpreted as the pricing kernel conditional upon being in state-s:

\[ \frac{d\Lambda^s(t)}{\Lambda^s(t)} = -r_s \, dt - \sigma \, dZ^s_0(t), \]  

(29)

where the state-contingent spot rates \( \{r_s\} \) are constants:

\[ r_s = \beta + \mu_s - \frac{\sigma^2}{2}. \]  

(30)

Using Ito’s lemma, we can formally integrate this as

\[ \Lambda^s(T) / \Lambda^s(0) = e^{-(r_s + \frac{\sigma^2}{2}) T - \sigma Z_s^1(T)}. \]  

(31)

Plugging this into equation (28) we can express the Arrow/Debreu prices as

\[ A(F_t | F_0) = \sum_s \pi_s(0) \xi_s(0) \pi(F_t | F_0, s) \left( \frac{\Lambda^s(F_t)}{\Lambda^s(F_0)} \right) \]  

\[ = \sum_s \pi^Q_s(t) \mathbb{E} \left[ \left( \frac{\Lambda^s(t)}{\Lambda^s(0)} \right) 1_{\{F_t < F_s\}} | F_0, s \right], \]  

(32)

where we have defined the risk adjusted probabilities

\[ \pi^Q_s(t) \equiv \pi_s(t) \xi_s(t). \]  

(33)

More generally, the date-t price \( V^{D(F_T)} \) of a security with contingent cash flows \( D(F_T) \) at date-T if state-\( F_T \) occurs is:

\[ V^{D(F_T)} = \sum_s \pi^Q_s(t) \mathbb{E} \left[ \left( \frac{\Lambda^s(T)}{\Lambda^s(t)} \right) D(T | F_t, s) \right] \]  

\[ = \sum_s \pi^Q_s(t) e^{-r_s (T-t)} \mathbb{E}^{Q_s} [D(T | F_t, s)], \]  

(34)

where \( dZ^Q_{1,s} = dZ_{1,s} + \sigma \, dt \), and all other Brownian motions orthogonal to \( dZ_{1,s} \) are unaffected by the change of measure. Hence, under the risk neutral measure we have

\[ dX(t) = [\psi^Q - \kappa X(t)] \, dt + \Sigma \, dW^Q(t), \]  

(35)

where the vector \( \psi^Q \) is defined via:

\[ \psi^Q = \psi - \sigma \Sigma \rho. \]  

(36)

Here, \( \rho \) is a vector of correlations with elements \( \rho_j = \text{Cor}(dW_j, dZ_0) \).
3.4 The Risk-free Zero-Coupon Bond

Using equation (34), the price of the zero-coupon bond that pays $D = 1$ unit of consumption in all states of nature at date-$T$ is:

$$P(\pi(t), T - t) = \sum_s \pi_s Q_s(t) e^{-r_s(T-t)} E^{Q_s}[1|{\mathcal F}_t, s]$$

$$= \sum_s \pi_s Q_s(t) e^{-r_s(T-t)}.$$  \hspace{1cm} (37)

3.5 Defaultable Zero-Coupon Bond

The price of the zero-coupon bond that pays $D = 1_{i > T}$ one unit of consumption at date-$T$ if the firm does not default by that date, and zero otherwise, is:

$$B^i(\pi(t), X(t), T - t) = \sum_s \pi_s Q_s(t) e^{-r_s(T-t)} E^{Q_s}[1_{\{i > T\}}|{\mathcal F}_t, s]$$

$$= \left[ \sum_{s=1}^M \pi_s Q_s(t) e^{-r_s(T-t)} B^i_s(X(t), T - t) \right] 1_{\{i > T\}},$$  \hspace{1cm} (38)

where we have defined

$$B^i_s(X(t), T - t) \equiv E^{Q_s}\left[ e^{-\int_t^T du \lambda_{i,s}(X(u))} |{\mathcal F}_t, s} \right].$$  \hspace{1cm} (39)

The reason that the price of the risky bond can be written as a weighted sum of terms, each of which can be expressed in a “reduced-form structure” is because, conditional on being in state-$s$, we are in a doubly stochastic framework.

The implication of equation (39) is that $e^{-\int_t^T du \lambda_{i,s}(X(u))} B^i_s(X(t), t, T)$ is a $Q$-martingale, implying that the solution for $B^i_s(X(t), t, T)$ satisfies the PDE (in this equation, we drop the $(i,s)$ subscripts on $B^i_s(X(t), t, T)$ and $\lambda_{i,s}$ to improve readability)

$$0 = -\lambda(X(t)) B + B_t + \sum_j B_j \left[ \psi^Q_j - \sum_m \kappa_{jm} X_m \right] + \frac{1}{2} \sum_{j,j'} B_{j,j'} \sum_m \Sigma_{jm} \Sigma_{j'm}.$$  \hspace{1cm} (40)

Here, we use the notation $B_i \equiv \frac{\partial}{\partial t} B_j \equiv \frac{\partial}{\partial X_j} B$, etc.

Given that $\lambda_{i,s}(X(u))$ is linear in the state vector $X(t)$ via equation (11) and that the risk-neutral dynamics are affine, it is well known that the solution to this expectation takes the form:

$$B^i_s(X(t), T - t) = e^{M_{i,s}(T-t)-N'_{i,s}(T-t) X(t)},$$  \hspace{1cm} (41)
where, as shown in the appendix, the deterministic coefficients satisfy
\[
N_{i,s}(\tau) = (\kappa')^{-1} \left[ J_n - \exp(-\kappa'\tau) \right] \beta_{i,s} \tag{42}
\]
\[
M_{i,s}(\tau) = \int_0^\tau du \left[ -\alpha_{i,s} - N'_{i,s}(u)\psi^Q + \frac{1}{2} N''_{i,s}(u)\Sigma'N_{i,s}(u) \right]. \tag{43}
\]

3.6 The risk-neutral survival probability

We note that the risk-neutral survival probability is obtained from the above expression by setting the risk-free rate component to zero. Specifically,
\[
S^i(\pi(t), X(t), T-t) = \sum_s \pi^Q_s(t) E^{Q_s} \left[ 1_{\{\tau_i > T\}} | \mathcal{F}_t, s \right] \tag{44}
\]
\[
= \left[ \sum_{i=1}^M \pi^Q_s(t) B^i_s(X(t), T-t) \right] 1_{\{\tau_i > t\}},
\]

3.7 Pricing kernel

Here we identify the stochastic discount factor \( \Lambda(\mathcal{F}_T) \) that determines the price of a generic asset \( V^D(\mathcal{F}_T)(\mathcal{F}_t) \) with state-contingent cash flows \( D(\mathcal{F}_T) \) via the equation:
\[
\Lambda(\mathcal{F}_t) V^D(\mathcal{F}_T)(\mathcal{F}_t) = \int d\mathcal{F}_T \pi(\mathcal{F}_T | \mathcal{F}_t) \Lambda(\mathcal{F}_T) D(\mathcal{F}_T). \tag{45}
\]

In order to do so, it is convenient to notionally write
\[
\frac{d\Lambda(t)}{\Lambda(t)} = -r(t) dt - \phi(t) dZ_\sigma(t) + \sum_{i=1}^n \Gamma_i(t) \left[ d1_{\{\tau_i < t\}} \right] \frac{d\lambda_p^i(t)}{\lambda(t)} \tag{46}
\]

Here, we identify the risk free rate \( r(t) \), the market prices of Brownian Motion risk \( \phi(t) \) and jump risk \( \Gamma_i(t) \). To do so, note that we can express \( \frac{\Delta(t + dt)}{\Delta(t)} = 1 + \frac{dt}{\Delta(t)} \), implying that we can express the price of the asset with cash flows \( D(\mathcal{F}_{t+dt}) \) paid out at time-(\( t + dt \)) as:
\[
V^D(\mathcal{F}_{t+dt})(\mathcal{F}_t) = \int d\mathcal{F}_{t+dt} \pi(\mathcal{F}_{t+dt} | \mathcal{F}_t) \times \left[ 1 - r(t) dt - \phi(t) dZ_\sigma(t) + \sum_{i=1}^n \Gamma_i(t) \left[ d1_{\{\tau_i < t\}} \right] \frac{d\lambda_p^i(t)}{\lambda(t)} \right] D(\mathcal{F}_{t+dt}). \tag{47}
\]

From equations (29) and (34), note that we can also express the security price as
\[
V^D(\mathcal{F}_{t+dt})(\mathcal{F}_t) = \sum_s \pi^Q_s(t) \int d\mathcal{F}_{t+dt} \pi(\mathcal{F}_{t+dt} | \mathcal{F}_t, s) \left[ 1 - r_s dt - \sigma dZ^s_\sigma(t) \right] D(\mathcal{F}_{t+dt}). \tag{48}
\]

We now use these two different expressions to identify \( r(t) \), \( \phi(t) \) and \( \Gamma_i(t) \). In particular we consider three securities:
1. Consider a risk-free security that pays $D(F_{t+dt}) = 1$ in all states of nature. Comparing equations (47) and (48), we find:

$$1 - r(t) dt = \sum_s \pi_s^Q(t) [1 - r_s dt],$$

implying that the risk-free rate $r(t)$ satisfies

$$r(t) = \sum_s \pi_s^Q(t) r_s.$$  

2. Consider a security that pays $D(F_{t+dt}) \equiv dZ_0 = dZ_0^s + \left( \frac{\mu_s - \mu^P(t)}{\sigma} \right) dt$, where we have used equation (15). Comparing equations (47) and (48), we find that the price of risk on $dZ_0$ is

$$-\phi dt = -\sigma dt + \sum_s \pi_s^Q(t) \left( \frac{\mu_s - \mu^P(t)}{\sigma} \right) dt \equiv -\sigma dt + \left( \bar{\pi}^Q(t) - \bar{\mu}^P(t) \right) dt,$$

where we have defined $\bar{\pi}^Q(t) = \sum_s \pi_s^Q(t) \mu_s$. Simplifying, we find

$$\phi = \sigma - \left( \bar{\pi}^Q(t) - \bar{\mu}^P(t) \right).$$

3. Finally, consider a security that pays $D(F_{t+dt}) = d1_{\{\tau_i < t\}}$. Comparing equations (47) and (48), we find:

$$\lambda^P_i(t) (1 + \Gamma_i(X_i)) dt = \sum_s \pi_s^Q(t) \lambda_{i,s}(X_i) dt.$$

Defining the risk-neutral intensity via

$$\tilde{\lambda}^Q_i(X_i) = \sum_s \pi_s^Q(t) \lambda_{i,s}(X_i),$$

we find

$$\Gamma_i(X_i) = \frac{\tilde{\lambda}^Q_i(X_i) - \lambda^P_i(X_i)}{\lambda^P_i(X_i)}.$$

The fact that $\Gamma_i(X_i)$ differs from zero implies that sovereign jumps-to-default are priced in this economy.
3.8 Expression for the sovereign CDS

We obtain the expression for the sovereign CDS from our discount bond. The CDS payment equates the present value of the fee leg given by:

\[
PvFee(c^i) = \sum_{j=1}^{n} B^i(\pi_t, X(t), t_j) c^i \Delta + \sum_{j=1}^{n} P(\pi_t, \frac{t_j + t_{j-1}}{2})(S^i(\pi_t, X(t), t_{j-1}) - S^i(\pi_t, X(t), t_j)) c^i \Delta \frac{\Delta}{2},
\]

where the second component is the present value of the accrued interest upon default (assumed to occur half-way between \(t_{j-1}\) and \(t_j\) for simplicity). Payments are made at pre-specified dates \(t = t_0, t_1, t_2, \ldots, t_n\). \(\Delta = t_j - t_{j-1}\) is the time between promised coupon payment (typically one quarter).

The present value of the contingent default payment leg is:

\[
PvDef = \sum_{j=1}^{n} P(\pi_t, \frac{t_j + t_{j-1}}{2})(S^i(\pi_t, X(t), t_{j-1}) - S^i(\pi_t, X(t), t_j)) L,
\]

(56)

where \(L\) is the expected loss given default experienced upon a sovereign default. For example, Pan Singleton (?) discuss the fact that market convention is to set \(L = 0.75\) for sovereign risk (as opposed to \(L = 0.4\) for corporate bonds). They do find that their maximum likelihood estimates are not too distinct for most countries from that market convention.

The fair credit default swap spread is the number \(c^i\) that sets

\[
PvFee(c^i) = PvDef
\]

4 Empirical Estimation

TO BE WRITTEN

4.1 Data

TO BE WRITTEN

4.2 Macroeconomic Conditions Indices for Euro Zone Countries

TO BE WRITTEN

4.3 Preliminary Linear Regressions

TO BE WRITTEN
4.4 Model Estimation and Results

TO BE WRITTEN

5 Conclusion

We investigate a general equilibrium framework that captures contagion risk in defaultable bonds. The model has two important ingredients: a hidden state of nature which impacts expected consumption growth and default probabilities, and a representative agent with fragile beliefs (Hansen and Sargent (2010)). Even though capturing contagion implies that our intensity-based model falls outside of the “doubly stochastic” framework, bond prices remain tractable, in turn facilitating empirical investigation. We apply this model to sovereign countries in the European Union (EU) and find that it matches well the historical time series and cross section of sovereign CDS spreads. In particular, the model can capture large and highly correlated credit spreads even when default probabilities and correlations in macroeconomic fundamentals are low. Finally, we also identify conditions for which the marginal utility of the agent with fragile beliefs generate time-consistent state prices.
References


Figure 1: **Sovereign CDS Spreads.** The plots show 5-year sovereign CDS spreads for Euro zone countries. The sample period is 2004/02/12-2010/12/10. Source: Markit Financial Information Services.
Figure 2: Macroeconomic Conditions Indices. The plots show the macroeconomic conditions indices for Euro zone countries. The sample period is 2001/01-2010/06/30.
Figure 3: Hidden State Probability. The plots show $P$- and $Q$-measure estimates for the probability that the economy is in the ‘good’ state. The sample period is 2004/02/12-2010/06/30.
Figure 4: **Sovereign CDS spreads: Model vs. Data.** The plots contrast model-implied 5-year sovereign CDS spreads with actual data. The sample period is 2004/02/12-2010/06/30.
Figure 5: Default Intensities. The plots show the $P$- and $Q$-measure estimates for the default intensities, $\lambda$ and $\lambda^Q$. 
Table 1: **Data Description.** The table shows data sources for each Euro zone country.

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<td>Direção Geral do Orçamento</td>
<td>Banco de España</td>
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Table 2: Preliminary OLS Regressions. For each Euro zone country, the table shows OLS regressions for the models, Model 1:  
\[ \text{CDS}_i = b_{0,i} + b_{1,i} \text{MCI}_i + b_{2,i} \text{MCI}_{EU\perp} + b_{3,i} \log \text{VIX} + b_{4,i} \text{BB-BBB spread} + b_{5,i} \text{PC}_{CDS_{-i}} + \varepsilon_i \]
Model 2:  
\[ \Delta \text{CDS}_i = b_{0,i} + b_{1,i} \Delta \text{MCI}_i + b_{2,i} \Delta \text{MCI}_{EU\perp} + b_{3,i} \Delta \log \text{VIX} + b_{4,i} \Delta \text{BB-BBB spread} + b_{5,i} \Delta \text{PC}_{CDS_{-i}} + \varepsilon_i. \]

The dependent CDS\(_i\) variable is the daily 5-year credit default swap basis-point spread for country \(i\). MCI\(_i\) is the macroeconomic conditions index for country \(i\). MCI\(_{EU\perp}\) is the residual of an OLS regression of the daily Euro zone MCI\(_{EU}\) against the daily country-\(i\) MCI\(_i\) and a constant. The log VIX variable is the logarithmic daily S&P 500 percentage implied volatility index published by the Chicago Board Option Exchange. The BB-BBB spread variable is the difference between Bank of America Merrill Lynch corporate bond effective percentage yield indices, sampled at the daily frequency. The PC\(_{CDS_{-i}}\) variable is the first principal component extracted from the panel of 5-year CDS spreads for Euro zone countries, excluding country \(i\), Ireland, and the Netherlands (we exclude Ireland and the Netherlands due to data availability).

Compared to Model 1, Model 2 has both left- and right-hand side variables in differences rather than in levels, e.g., we measure \(\Delta \text{CDS}_i\) with the series of daily overlapping changes in the 5-year CDS spreads for country \(i\) relative to the prior month, \(\Delta \text{CDS}_i(t) = \text{CDS}_i(t) - \text{CDS}_i(t-21)\). For Ireland, the sample period is 2004/05-2010/06; for the Netherlands it is 2006/06-2010/06; for the other countries it is 2004/02-2010/06. The coefficient \(t\)-ratios in brackets are based on (Newey-West) heteroskedasticity and autocorrelation robust standard errors. The Local Ratio is the adjusted \(R^2\) for the regression that has only the local and regional macroeconomic conditions indices MCI\(_i\) and MCI\(_{EU\perp}\) among the explanatory variables, divided by the adjusted \(R^2\) of each regression.

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<td>16.89</td>
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<tr>
<td>( 4.97)</td>
<td>( 1.43)</td>
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| \( PC_{CDS,i}^1 \) | 1.26 | 1.12 |
| ( 13.04) | ( 6.14) |       |

| Adj. R^2     | 0.68 | 0.01 |
| Local Ratio  | 1.13 | 5.63 |
| mean(abs(\( \epsilon \)) | 59.78 | 27.78 |
| max(abs(\( \epsilon \)) | 552.71 | 475.26 |

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<td>( 0.21)</td>
<td>-0.08</td>
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<td>( 4.31)</td>
<td>( 2.14)</td>
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| \( PC_{CDS,i}^1 \) | 0.30 | 0.31 |
| ( 9.32) | ( 4.79) |       |

| Adj. R^2     | 0.76 | -0.00 |
| Local Ratio  | 1.02 | 1.84 |
| mean(abs(\( \epsilon \)) | 24.56 | 16.51 |
| max(abs(\( \epsilon \)) | 243.65 | 161.22 |
Table 2, continued

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<td>0.22 0.26</td>
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<td>6.58 6.58</td>
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</tr>
<tr>
<td>max(abs($\varepsilon$))</td>
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<td>54.06 54.54</td>
<td></td>
</tr>
</tbody>
</table>
### Panel J: Portugal

|                  | Model 1 | Model 2 | Model 1 | Model 2 | Model 1 | Model 2 | Model 1 | Model 2 | Model 1 | Model 2 | Model 1 | Model 2 | Model 1 | Model 2 | Model 1 | Model 2 | Model 1 | Model 2 | Model 1 | Model 2 |
|------------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $MCI_i$          | 1.31    | 0.69    | -1.16   | -1.68   | -0.51   | -1.03   | -1.28   | -1.52   | -1.48   | -0.36   | -1.17   | -1.84   | -2.21   | -2.08   | -1.30   |         |         |         |         |
|                  | (1.02)  | (1.08)  | (1.32)  | (1.80)  | (5.16)  | (-1.17) | (-1.84) | (-2.21) | (-2.08) | (-1.30) |         |         |         |         |         |         |         |         |         |         |
| $MCI_{EU_i}$     | 2.71    | 1.24    | 1.59    | 0.00    | 0.05    | 0.04    | 0.01    | -0.11   | 0.11    | 0.03    |         |         |         |         |         |         |         |         |         |         |
|                  | (1.85)  | (1.19)  | (1.56)  | (0.04)  | (1.15)  | (0.11)  | (0.03)  |         | (0.11)  | (0.03)  |         |         |         |         |         |         |         |         |         |         |
| log $VIX$        | 79.85   | 95.36   | 38.95   | 9.10    | 41.30   | 44.32   | 39.06   | 14.74   |         |         |         |         |         |         |         |         |         |         |         |         |
|                  | (7.44)  | (4.24)  | (1.36)  | (3.34)  | (3.29)  | (3.44)  | (2.46)  | (2.45)  |         |         |         |         |         |         |         |         |         |         |         |         |
| BB-BBB spread    | 28.42   | 4.37    |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |
|                  | (2.34)  |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |
| $PC_{CDS,-1}$    |         |         | 0.36    |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |
|                  |         |         | (2.31)  |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |
| Adj. R²          | 0.01    | 0.10    | 0.27    | 0.30    | 0.34    | 0.98    | 0.01    | 0.04    | 0.10    | 0.16    | 0.16    | 0.16    | 0.78    |         |         |         |         |         |         |
|                  | (23.51) |         |         |         |         |         | (7.52)  |         |         |         |         |         |         |         |         |         |         |         |         |         |
| Local Ratio      | 7.03    | 1.00    | 0.36    | 0.33    | 0.29    | 0.10    | 2.64    | 1.00    | 0.35    | 0.23    | 0.23    | 0.05    |         |         |         |         |         |         |         |         |
| mean(abs($\varepsilon$)) | 41.38  | 39.61   | 31.54   | 34.22   | 31.82   | 5.29    | 12.71   | 12.96   | 13.34   | 13.47   | 13.09   | 6.09    |         |         |         |         |         |         |         |         |
| max(abs($\varepsilon$)) | 429.24 | 413.49  | 375.07  | 346.04  | 353.17  | 93.53   | 295.36  | 294.67  | 266.43  | 265.00  | 266.67  | 99.54   |         |         |         |         |         |         |         |         |

### Panel K: Spain

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Table 3: **CDS Pricing Errors: Model vs. OLS Regressions.** The table compares 5-year CDS spreads pricing errors from the model to those from the OLS regression in Table 2.

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<th>OLS, with country specific MCI</th>
<th>OLS, with MCI, fin. vars., $PC_{CDS-i}^{t}$</th>
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<tr>
<td>mean abs. err.</td>
<td>2.63</td>
<td>12.92</td>
<td>3.33</td>
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<td>max abs. err.</td>
<td>28.31</td>
<td>84.78</td>
<td>36.09</td>
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<td><strong>Panel B: Germany</strong></td>
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<td>mean abs. err.</td>
<td>2.07</td>
<td>9.37</td>
<td>3.74</td>
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<td>23.01</td>
<td>55.85</td>
<td>40.51</td>
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<td><strong>Panel C: Greece</strong></td>
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<td>2.66</td>
<td>59.78</td>
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<td>21.46</td>
<td>552.71</td>
<td>244.91</td>
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<td><strong>Panel D: Italy</strong></td>
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<td>3.86</td>
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<td>8.76</td>
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<td>max abs. err.</td>
<td>58.65</td>
<td>230.62</td>
<td>68.43</td>
</tr>
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</table>
7 Appendix

7.1 Proof that Robust and fragile Preferences are not Time-Consistent

Here we demonstrate that equilibrium state prices in an economy where the representative has preferences for both robustness and fragility are not time-consistent. Define \( A(F_T | F_t) \) as the date-\( t \) price of an Arrow-Debreu security that pays $1 at date-\( T \) iff state \( F_T \) occurs. Here we examine the necessary conditions for the following formula to hold:

\[
A(\uparrow \uparrow | 0) = A(\uparrow | \uparrow) A(\uparrow | 0).
\]  (57)

Intuitively, the LHS is the date-0 price of a security that pays $1 at date-2 if \( F_2 = \uparrow \uparrow \). In contrast, the RHS can be thought of as an agent who purchases, at date-0, \( A(\uparrow | \uparrow) \) shares of the A/D security that pays at $1 at date-1 if \( F_1 = \uparrow \). Note that, if \( \uparrow \) occurs, such a strategy would pay off \( A(\uparrow | \uparrow) \), which is just enough at date-1 in state-\( \uparrow \) to purchase one share of the A/D security that pays $1 if \( F_2 = \uparrow \uparrow \) occurs. Thus, both sides of equation 1 imply two different methods of obtaining $1 at date-2 iff the \( \uparrow \uparrow \) state occurs, and hence, by the law of one price, the present value of these two portfolios should be the same.

7.1.1 The Economy

Consider an infinite period, discrete time economy where the exogenously specified dividend, which in equilibrium equals consumption, follows the binomial process:

\[
\log C(t + \Delta t) = \begin{cases} 
\log C(t) + x & \text{if } \uparrow \text{ occurs} \\
\log C(t) - x & \text{if } \downarrow \text{ occurs}
\end{cases}
\]  (58)

The agent is uncertain which of \( s \in (1, S) \) states the economy is in, but she has priors \( \pi(s | F_t) \equiv \pi_s(t) \). Conditional upon being in state-\( s \), the probability of an up state is \( \pi(\uparrow | s) \), and this probability is time-invariant in that \( \pi(\uparrow | s, F_t) = \pi(\uparrow | s) \forall F_t \).

Standard Bayesian updating implies that, conditional upon observing an \( \uparrow \) event at date-\( (t + 1) \), the probability that the economy is in state-\( s \) updates to:

\[
\pi_s(\uparrow) = \frac{\pi(s | F_{t+1} \cup \uparrow)}{\pi(\uparrow | F_{t+1})} = \frac{\pi(\uparrow | s) \pi(s | F_t)}{\sum_{s'} \pi(\uparrow | s') \pi(s' | F_t)} = \frac{\pi(\uparrow | s) \pi_s(t)}{\sum_{s'} \pi(\uparrow | s') \pi_{s'}(t)}. \]  (59)

7.1.2 Preference for Robustness

Here, we first assume that the economy is known to be in state-\( s \). Start with the equation

\[
V(t | s) = (1 - e^{-\beta dt}) \log C(t) + e^{-\beta dt} \min_{\xi > 0, E[\xi] = 1} \mathbb{E}_t [\xi V(t + dt | s) + \zeta_t \xi \log \xi | s] \]  (60)
The Lagrangian for this constrained minimization is:

\[ \mathcal{L} = \pi(\uparrow|s) [\xi(\uparrow|s)V(\uparrow|s) + \zeta_1 \xi(\uparrow|s) \log \xi(\uparrow|s)] \\
+ \pi(\downarrow|s) [\xi(\downarrow|s)V(\downarrow|s) + \zeta_1 \xi(\downarrow|s) \log \xi(\downarrow|s)] + \lambda [1 - \pi(\uparrow|s)\xi(\uparrow|s) - \pi(\downarrow|s)\xi(\downarrow|s)]. \]

The first order condition gives:

\[ \frac{\partial \mathcal{L}}{\partial \xi(\uparrow|s)} : 0 = \pi(\uparrow|s) [V(\uparrow|s) + \zeta_1 \log \xi(\uparrow|s) + \zeta_1 - \lambda] \]  

(61)

implying that

\[ \xi(\uparrow|s) = \exp \left( \frac{\lambda - \zeta_1}{\zeta_1} - \frac{V(\uparrow|s)}{\zeta_1} \right) \]  

(62)

To identify \( \lambda \), we plug back into constraint that \( E[\xi] = 1 \) to find

\[ \xi(\uparrow|s) = \frac{e^{-V(\uparrow|s)/\zeta_1}}{E[e^{-V(t+\delta t)/\zeta_1}]} = \frac{e^{-V(\uparrow|s)/\zeta_1}}{\pi(\uparrow|s) e^{-V(\uparrow|s)/\zeta_1} + \pi(\downarrow|s) e^{-V(\downarrow|s)/\zeta_1}}, \]  

(63)

with an analogous equation for \( \xi(\downarrow|s) \).

Plugging this back into the original equation, we find

\[ V(t|s) = (1 - e^{-\beta dt}) \log C(t) - \zeta_1 e^{-\beta dt} \log \left[ \pi(\uparrow|s) e^{-V(\uparrow|s)/\zeta_1} + \pi(\downarrow|s) e^{-V(\downarrow|s)/\zeta_1} \right]. \]  

(64)

Recursively, we also find\(^\text{13}\)

\[ V(\uparrow|s) = (1 - e^{-\beta dt}) \log C(\uparrow) - \zeta_1 e^{-\beta dt} \log \left[ \pi(\uparrow|s) e^{-V(\uparrow|s)/\zeta_1} + \pi(\downarrow|s) e^{-V(\downarrow|s)/\zeta_1} \right]. \]

\[ V(\uparrow\uparrow|s) = (1 - e^{-\beta dt}) \log C(\uparrow\uparrow) + \ldots \]  

(65)

For what we do below, it is worth noting that

\[ \frac{\partial V(t|s)}{\partial V(\uparrow|s)} = e^{-\beta dt} \pi(\uparrow|s) \xi(\uparrow|s). \]  

(66)

In particular, note that the RHS is independent of date-\( t \). The interpretation of this result is that, conditional upon being in a state-\( s \), things work as they would in a Black-Scholes binomial tree, with, e.g., \( \pi(\uparrow\uparrow|s,0) = [\pi(\uparrow|s,0)]^2 \) and \( A(\uparrow\uparrow|s,0) = [A(\uparrow|s,0)]^2 \).

### 7.1.3 Fragile Beliefs

The agent must still decide how to weight the different states. We assume she does this maximizing the following objective:

\[ V(t) = \sum_{s=1}^{m} \pi_s(t) [\xi_s(t)V(t|s) + \zeta_2 \xi_s(t) \log \xi_s(t)]. \]  

(67)

\(^{13}\)This is a slight abuse of notation: \( V(\uparrow|s) \) should really be written as \( V(F_{t+1} = (F_t \cup \uparrow)|s) \).
subject to the constraint

\[ 0 = \lambda \left( 1 - \sum_{s=1}^{m} \pi_s(t) \xi_s(t) \right) \]  

(68)

Setting up the Lagrangian, the first order condition gives

\[ \frac{\partial}{\partial \xi_s} : 0 = \pi_s(t) [V(t|s) + \zeta_2 \log \xi_s(t) + \zeta_2 - \lambda] \]  

(69)

Using the expectation constraint to identify \( \lambda \), we find

\[ \xi_s(t) = e^{-V(t|s)/\zeta_2} \sum_{s'} \pi_{s'}(t) e^{-V(t|s')/\zeta_2} \]  

(70)

Elsewhere, we have shown that

\[ V(t|s) = \log C(t) + B_s \]  

(71)

Plugging this in, we can write

\[ \xi_s(t) = e^{-B_s/\zeta_2} \sum_{s'} \pi_{s'}(t) e^{-B_{s'}/\zeta_2} \]  

(72)

It is worth noting that \( \xi_s(t) \) depends on date-\( t \) only through the \( \{\pi_s(t)\} \). Therefore, with slight abuse of notation, we have

\[ \xi_s(t) = e^{-B_s/\zeta_2} \sum_{s'} \pi_{s'}(t) e^{-B_{s'}/\zeta_2} \]  

(73)

In what follows, it is worth noting from equation (59) that

\[ \pi_s(\uparrow) \xi_s(\uparrow) = \frac{\left( \pi(\uparrow|s) \pi_s(t) \right)}{\pi(\uparrow|\pi_s)} e^{-B_s/\zeta_2} \sum_{s'} \pi_{s'}(t) e^{-B_{s'}/\zeta_2} \]  

(74)

The fact that both the numerator and denominator are linear in \( \pi_s^Q(t) \equiv \pi_s(t) \xi_s(t) \) is important in what follows and implies that fragility is well-specified.

Getting back to the issue at hand and plugging equation (70) into equation (67), we find

\[ V(t) = -\zeta_2 \log \left( \sum \pi_s(t) e^{-V(t|s)/\zeta_2} \right) \],  

(75)

where, as noted above

\[ V(t|S) = \left( 1 - e^{-\beta dt} \right) \log C(t) - \zeta_1 e^{-\beta dt} \log \left[ \pi(\uparrow|s) e^{-V(\uparrow|s)/\zeta_1} + \pi(\downarrow|s) e^{-V(\downarrow|s)/\zeta_1} \right] \]  

(76)

In what follows, it is worth noting that

\[ \frac{\partial V(t)}{\partial \xi_s(t)} = \pi_s(t) \xi_s(t) \]  

(77)
7.2 Arrow Debreu Prices

To identify the Arrow-Debreu prices, we consider the agent starting at the optimal controls, and then considering modifying those controls by purchasing $\epsilon$ shares of the Arrow/Debreu security that pays $1(F_{\uparrow} = \uparrow)$. As such, her current consumption drops by $\epsilon \pi_{\uparrow} | 0 \rangle$, and her date-1 consumption in the $\uparrow$ state increases by $\epsilon$, with all other consumption in time and event space held constant:

$$[C(t), C(\uparrow)] \Rightarrow [C(t) - \epsilon \pi_{\uparrow} | 0 \rangle, C(\uparrow) + \epsilon] \quad (78)$$

Such an infinitesimal change has no effect on optimal utility:

$$0 = \delta V(t)$$

$$= \sum_{s} \pi_{s}(t) \xi_{s}(t) \delta V(t|s)$$

$$= \sum_{s} \pi_{s}(t) \xi_{s}(t) \left[ (1 - e^{-\beta dt}) \left( \frac{1}{C_{\uparrow}} \right) (-\epsilon \pi_{\uparrow} | 0 \rangle) + e^{-\beta dt} \pi(\uparrow | s) \xi(\uparrow | s) \left( 1 - e^{-\beta dt} \right) \left( \frac{1}{C_{\uparrow}} \right) (\epsilon) \right],$$

where equation (79) comes from equation (77). The solution to this is

$$A(\uparrow | 0) = \left( \frac{C_{\uparrow}}{C_{\uparrow}} \right) e^{-\beta dt} \sum_{s} \pi_{s}(t) \xi_{s}(t) \pi(\uparrow | s) \xi(\uparrow | s)$$

$$= e^{-x} e^{-\beta dt} \sum_{s} \pi_{s}(t) \xi_{s}(t) \pi(\uparrow | s) \xi(\uparrow | s). \quad (80)$$

It is important to note that the only $t$-dependence on the right hand side is through the $\pi_{s}(t) \xi_{s}(t)$. In particular, if at date-1 an $\uparrow$-state occurs, the price the A/D security that pays $1(F_{\uparrow\uparrow} = \uparrow\uparrow)$ (with some abuse of notation) is:

$$A(\uparrow\uparrow | \uparrow) = e^{-x} e^{-\beta dt} \sum_{s} \pi_{s}(F_{\uparrow} \cup \uparrow) \xi_{s}(F_{\uparrow} \cup \uparrow) \pi(\uparrow | s) \xi(\uparrow | s)$$

$$\equiv e^{-x} e^{-\beta dt} \sum_{s} \pi_{s}(\uparrow) \xi_{s}(\uparrow) \pi(\uparrow | s) \xi(\uparrow | s)$$

$$= e^{-x} e^{-\beta dt} \sum_{s} \frac{\pi(\uparrow | s) \pi_{s}(t) \xi_{s}(t) \pi(\uparrow | s)}{\sum_{s'} \pi_{s'}(t) \xi_{s'}(t) \pi(\uparrow | s')} \xi(\uparrow | s) \xi(\uparrow | s)$$

$$= e^{-x} e^{-\beta dt} \sum_{s} \pi_{s}(t) \xi_{s}(t) \pi(\uparrow | s) \xi(\uparrow | s), \quad (81)$$

where we have used equation (74) in the last line.

Finally, consider the two-period infinitesimal variation:

$$[C(t), C(\uparrow\uparrow)] \Rightarrow [C(t) - \epsilon A(\uparrow\uparrow | 0), C(\uparrow\uparrow) + \epsilon] \quad (82)$$

We find

$$A(\uparrow\uparrow | 0) = e^{-2x} e^{-2\beta dt} \sum_{s} \pi_{s}(t) \xi_{s}(t) \pi^{2}(\uparrow | s) \xi^{2}(\uparrow | s). \quad (83)$$
7.3 Time Consistency

The LHS of equation 1 is equation (83). Combining equations (80) and (81), we find the RHS of equation 1 is

\[ A(\uparrow \uparrow | \uparrow)A(\uparrow | 0) = \left[e^{-\beta t} \sum_s \pi_s(t) \xi_s(t) \pi^2(\uparrow | s) \xi(\uparrow | s) \right] \left[e^{-\beta t} \sum_s \pi_s(t) \xi_s(t) \pi(\uparrow | s') \right]. \]

Dividing both sides by \(e^{-2\beta t}\) and multiplying both sides by \(\sum_{s'} \pi_{s'}(t) \xi_{s'}(t) \pi(\uparrow | s')\), we obtain

\[ "LHS" = \left[\sum_s \pi_s(t) \xi_s(t) \pi^2(\uparrow | s) \xi^2(\uparrow | s) \right] \left[\sum_s \pi_s(t) \xi_s(t) \pi(\uparrow | s) \xi(\uparrow | s) \right] \]

\[ "RHS" = \left[\sum_s \pi_s(t) \xi_s(t) \pi^2(\uparrow | s) \xi(\uparrow | s) \right] \left[\sum_s \pi_s(t) \xi_s(t) \pi(\uparrow | s) \xi(\uparrow | s) \right] \]

(84)

Now, since the priors \(\{\pi_s(t)\}\) are completely arbitrary (except that they sum to unity), there are only two ways the LHS will equal the RHS:

- There is no hidden state. That is, \(\pi_s(t) = 1\) for one value of \(s\) and zero for all others.
- \(\xi(\uparrow | s) = 1\), which implies that preference for robustness has been ‘turned off’

The first approach captures the known result that recursive preferences are time-consistent. The second condition implies that a combination of fragility and conditional time-separable preferences may be time-consistent in this framework. We prove this below.

7.4 Proof that Fragile Beliefs Economy is Time consistent

Our candidate pricing kernel has following dynamics:

\[ \frac{d\Lambda(t)}{\Lambda(t)} = -r(t) dt - \sum_{k=0}^{K} \phi_k(t) dZ_k(t) - \sum_i \frac{\lambda^Q_i - \lambda^P_i}{\lambda^P_i} dM_i(t) \]

(85)

Note that the pricing kernel must be written in the filtration of the agent (in other words, \(Z_0(t)\) is a Brownian motion and \(N_i(t)\) has intensity \(\lambda_i(t)\)). Further, in (109) below we show that the market price of Brownian risk is given by:

\[ \phi_k = \sigma_k 1_{k=0} - \frac{\mu^Q_k - \mu^P_k}{\sigma_k} \]

(86)

\[ \mu^Q_k = \sum_s \pi^Q_s H_{k,s} \]

(87)

\[ \mu^P_k = \sum_s \pi^P_s H_{k,s} \]

(88)
Now, according to our calculations,

\[ r(t) = \sum_s \pi^Q_s(t) r^s \]  
\[ \bar{X}_i^p = \sum_s \pi_s(t) \lambda_i(t) \]  
\[ \bar{X}_i^q = \sum_s \pi^Q_s(t) \lambda_i(t) \]  
\[ \pi^Q_s(t) = \frac{\chi_s \pi_s(t)}{\sum_s \chi_s \pi_s(t)} \]  
\[ d\pi_s(t) = \pi_s(t) \sum_{k=0}^K \frac{\mu_{k,s} - \Pi_k^P(t)}{\sigma_k} dZ_k(t) + \sum_{i=1}^N \alpha_i(t^-) dM_i(t). \]

### 7.4.1 Pricing the risk-free bond

Here we consider the case of the risk-free bond first. We want to show that the value of a zero-coupon bond if priced as the gradient of the agent with fragile beliefs agrees with the price in the economy where prices are determined by the pricing kernel of equation (85). In other words we want to show that:

\[ E_t[\Lambda(T) \Lambda(t)] = \sum_s \pi^Q_s(t) E_t[\Lambda^s(T)|s, F_t]. \]

Alternatively, since:

\[ E_t[\Lambda(T) \Lambda(t)] = E_t[\sum_s \pi^Q_s(t)] \]
\[ = E_t[\sum_s \pi^Q_s(t)] \]
\[ = E_t^Q[\sum_s \pi^Q_s(t)] \]

and since

\[ E_t[\Lambda^s(T)|s, F_t] = E_t^Q[e^{-r^s(T-t)}|F_t, s], \]

we need to show that:

\[ E_t^Q[e^{-\int_t^T \sum_s \pi^Q_s(u)r^s du}] = \sum_s \pi^Q_s(t) E_t^Q[e^{-r^s(T-t)}|F_t, s]. \]

To prove this, it is sufficient to show that \( M(t) \) defined as:

\[ M(t) = e^{-\int_t^T \sum_s \pi^Q_s(u)r^s du} \sum_s \pi^Q_s(t) e^{-r^s(T-t)} \]

is a Q-martingale.

Applying Itô to \( M(t) \) we find as above:

\[ dM(t) = e^{-\int_0^t \sum_s \pi^Q_s(u)r^s du} \sum_s \pi^Q_s(t) e^{-r^s(T-t)} \left\{ - \sum_{s'} \pi^Q_{s'} r^{s'} dt + r^s dt + \frac{d\pi^Q_s(t)}{\pi^Q_s(t)} \right\} \]
Therefore we see that a sufficient condition for $E^Q[dM(t)] = 0$ is that

$$E^Q \{ \frac{d\pi_s^Q(t)}{\pi_s^Q(t)} \} = \left( \sum_{s'} \pi_s^Q r_s' - r_s \right) dt. \quad (101)$$

Since in our equilibrium we have

$$r_s = constant + \mu_{0,s}, \quad (102)$$

we see that necessary and sufficient condition for $E^Q[dM(t)] = 0$ is

$$E^Q \{ \frac{d\pi_s^Q(t)}{\pi_s^Q(t)} \} = \left( \sum_{s'} \pi_s^Q \mu_{0,s'} - \mu_{0,s} \right) dt \quad (103)$$

It is worth noting that this also shows that we cannot freely parameterize $r_s$ independent of the updating equation. That is, even with ‘robustness’ turned off and the conditional preferences are modeled as time-separable, in order for the fragile beliefs utility to be time-consistent, additional restrictions on the model are required.

To determine the appropriate market price of risk for our economy, here we derive the dynamics of $\pi^Q$ and see what the restriction (103) implies for $\phi$.

### 7.4.2 Dynamics of $\pi^Q$

Recall $\pi_s^Q(t) = \frac{\chi_s \pi_s(t)}{\sum_s \chi_s \pi_s(t)}$ where $\chi_s$ are constants. Also recall that

$$d\pi_s(t) = \pi_s(t) \sum_{k=0}^K \frac{\mu_{k,s} - \overline{\mu}_k^P(t)}{\sigma_k} dZ_k(t) + \sum_{i=1}^N \alpha_i(t^-) dM_i(t). \quad (104)$$

For simplicity, here we set $N = 1$ and drop the $i$-subscripts in the following. We then give the general result below, which follows trivially. We find

$$\frac{d\pi_s^Q(t)}{\pi_s^Q(t)} = \frac{d\pi_s^Q(t)}{\pi_s(t)} - \sum_s \chi_s d\pi_s^Q(t) + \frac{\left( \sum_s \chi_s d\pi_s^Q(t) \right)^2}{\sum_s \chi_s \pi_s(t)} \frac{d\pi_s^Q(t)}{\pi_s(t)} \sum_s \chi_s d\pi_s(t) + \frac{1}{\pi_s^Q} \left( \sum_s \chi_s \pi_s \frac{\lambda_s}{\pi_s^P} \right) - 1 dN_t,$$

where the superscript ‘c’ denotes the continuous part. Note that the jump component simplifies to $\left( \frac{\lambda_s}{\lambda^P} - 1 \right) dN_t$.

Now we determine the dynamics of $\sum_s \chi_s \pi_s(t)$:

$$\sum_s \chi_s d\pi_s(t) = \sum_{k=0}^K \frac{(\overline{\mu}_k^Q - \overline{\mu}_k^P)}{\sigma_k} dZ_k(t)^P + \left( \frac{\chi^Q}{\lambda^P} - 1 \right) (dN_t - \lambda^P dt). \quad (105)$$

Therefore, plugging in and simplifying we find:

$$\frac{d\pi_s^Q(t)}{\pi_s^Q(t)} = \sum_{k=0}^K \frac{\mu_{k,s} - \overline{\mu}_k^Q}{\sigma_k} dZ_k(t)^P - \frac{\overline{\mu}_k^Q - \overline{\mu}_k^P}{\sigma_k} dM_k(t) + \lambda_s - \chi^Q dt \left( dN_t - \lambda^Q dt \right). \quad (106)$$
Note that, interestingly, the jump component of $\pi^Q$ is a $Q$-martingale. However, under $Q$ the drift of $\pi^Q$ is in general not equal to zero. In fact, since the market price of Brownian risk is $\phi_k$ we have the following:

$$E^Q[\frac{d\pi^Q_s(t)}{\pi^Q_s(t)}] = -\sum_{k=0}^{K} \frac{\mu_{k,s} - \overline{\mu}^Q_k}{\sigma_k}(\phi_k + \frac{\mu^Q_k - \mu^P_k}{\sigma_k}) dt.$$  \hspace{1cm} (107)

Recall that we want to find the market price of risk $\phi$ such that equation (103) holds, i.e., such that:

$$E^Q[\frac{d\pi^Q_s(t)}{\pi^Q_s(t)}] = \overline{\mu}^Q_0 - \mu_{0,s}.$$  \hspace{1cm} (108)

Combining this with (107) above, we obtain the restriction:

$$\sigma_k 1_{\{k=0\}} = \phi_k + \overline{\mu}^Q_k - \overline{\mu}^P_k \quad \forall k = 0, 1, \ldots, K$$  \hspace{1cm} (109)

In turn, the $Q$ dynamics of $\pi^Q$ are given by

$$\frac{d\pi^Q_s(t)}{\pi^Q_s(t)} = (\overline{\mu}^Q_0 - \mu_{0,s}) dt + \sum_{k=0}^{K} \frac{\mu_{k,s} - \overline{\mu}^Q_k}{\sigma_k} dZ^Q_k(t) + \frac{\lambda_s - \overline{\lambda}^Q}{\overline{\lambda}^Q} (dB_s - \overline{X}^Q dt).$$  \hspace{1cm} (110)

### 7.4.3 Pricing an arbitrary contingent claim

The last section proved that the price of the risk-free bond is consistent with the market price dynamics implied by our pricing kernel given in (85) above. More generally, here we show that the price of any arbitrary contingent claim priced of the fragile beliefs agent’s marginal utility is consistent with the prices given by the economy with this pricing kernel. In other words, the fragile beliefs economy is time consistent (this follows naturally from the fact that the existence of a pricing kernel rules out arbitrage opportunities in the set of self-financing trading strategies subject to mild integrability conditions).

Consider a claim to an arbitrary $\mathcal{F}_T$ measurable payoff $X_T$. We want to show that the value of the payoff in the economy where prices are determined by the pricing kernel of equation (85) agrees with the price of the claim evaluated off of the gradient of the fragile beliefs agent. In other words we want to show that:

$$E_t[\frac{\Lambda(T)}{\Lambda(t)} X_T] = \sum_s \pi^Q_s(t) E[\Lambda^s(T) X_T|s].$$  \hspace{1cm} (111)

Alternatively, since:

$$E_t[\frac{\Lambda(T)}{\Lambda(t)} X_T] = E_t[\exp\left(-\int_t^T r(u)du\right) X_T]$$  \hspace{1cm} (122)

$$= E_t^Q[\exp\left(-\int_t^T \sum_s \sigma^2(u) r_s du\right) X_T],$$  \hspace{1cm} (113)

and since

$$E[\Lambda^s(T) X_T|s] = E^Q_t[\exp(-r^*(T-t)) X_T|\mathcal{F}_t, s],$$  \hspace{1cm} (114)
we need to show that
\[ E^Q_t[e^{\int_0^T \sum_s \pi_s^Q(u) r^s du} X_T] = \sum_s \pi_s^Q(t) E^{Q_s}[e^{r^s(T-t)} X_T | \mathcal{F}_t, s]. \] (115)

It suffices to verify that \( M(t) \) defined as:
\[ M(t) = e^{\int_0^t \sum_s \pi_s^Q(u) r^s du} \sum_s \pi_s^Q(t) E^{Q_s}[e^{r^s(T-t)} X_T | \mathcal{F}_t, s] \] (116)
is a \( Q \)-martingale. Note that from the definition of the \( Q_s \) measure, using the martingale representation theorem, there exists an \( \{ \mathcal{F}_t \} \) adapted process \( \psi(t) \) so that
\[ H_t := E^{Q_s}[X_T | \mathcal{F}_t, s] = E^{Q_s}[X_T | \mathcal{F}_0, s] + \int_0^t \left( \sum^{K} \psi_k(u) dZ_k^{Q_s}(u) + \psi_m(u) dm(u) \right). \] (117)
where \( dm(u) \) is a continuous martingale orthogonal to all \( Z_k^s \). Applying Itô to \( M(t) \) we find:
\[ dM(t) = e^{\int_0^t \sum_s \pi_s^Q(u) r^s du} \sum_s \pi_s^Q(t) e^{-r^s(T-t)} \left\{ -H_t \left( \sum_{s'} \pi_{s'}^Q r^{s'} dt + r^s dt + \frac{d\pi^Q(t)}{\pi^Q(t)} \right) + \sum_k \psi_k(t) dZ_k^{Q_s}(t) + \psi_m(t) dm(t) + \sum_k \psi_k(t) \left( \frac{\mu_{k,s} - \mu^Q_k}{\sigma_k} \right) dt \right\}. \] (118)

Now, since by definition of the risk-neutral measure we have
\[ dZ_k^Q(t) = dZ_k(t) + (\sigma_k \mathbf{1}_{\{k=0\}} - \frac{\mu^Q_k - \mu^P_k}{\sigma_k}) dt. \] (119)
Further, by definition of the \( Q_s \) measure we have:
\[ dZ_k^{Q_s}(t) = dZ_k(t) + \sigma_k \mathbf{1}_{\{k=0\}} dt. \] (120)
Finally, by definition
\[ dZ_k(t) = dZ_k(t) + \frac{\mu_{k,s} - \mu^P_k}{\sigma_k} dt. \] (121)
Combining, we obtain:
\[ dZ_k^{Q_s}(t) + \frac{\mu_{k,s} - \mu^Q_k}{\sigma_k} dt = dZ_k^Q(t). \] (122)
Thus
\[ E^Q_t \left[ \sum_k \psi_k(t) dZ_k^{Q_s}(t) + \sum_k \psi_k(t) \frac{\mu_{k,s} - \mu^Q_k}{\sigma_k} dt \right] = 0. \] (123)
Further, we note that since, \( m(t) \) is orthogonal to \( Z_k^s \), it is also orthogonal to all \( Z_k \), and therefore a martingale under \( Q \).

Now, we have proved previously that
\[ E_t^Q \left[ \frac{d\pi^Q_s(t)}{\pi^Q_s(t)} \right] = \left( \sum_{s'} \pi_{s'}^Q r^{s'} - r^s \right) dt. \] (124)
Therefore we have indeed proved that
\[ E^Q_t[dM(t)] = 0. \] (125)
This implies that we have determined a valid pricing kernel for our economy.